Surrey MiniSchool: "Methods of Direct Nuclear Reactions"

Lecture notes

Ian Thompson
Nuclear Theory and Modeling,
Lawrence Livermore National Laboratory, L-414, Livermore, CA 94551

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Thursday 8th (14.00-15:30):
Distorted Wave Theories Use of perturbation theory for elastic scattering estimates using Born and DW Born approximation (to introduce ideas of T-matrix, rather than SE solution, methods) Nonelastic scattering using perturbation theory (DWBA).

Friday 9th (09.30-11.00):
Production of the Compound Nucleus: Fusion reactions, barrier penetration concepts, transmission coefficients, Bohr-Wheeler equation.

Decay of the Compound Nucleus: The Hauser-Feshbach theory.

Material in part from the forthcoming book *Nuclear Reactions for Astrophysics* by I.J. Thompson and F. Nunes, to be published in 2009 by Cambridge University Press.

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2 Compound Nucleus Reactions

2.1 Introduction

Two nuclei may approach each other and fuse together. The first factor governing this process is clearly the penetration of any Coulomb or centrifugal barrier present at middle distances. If the scattering wave comes in from large relative distances, then it will be attenuated by the time it has tunnel through any Coulomb barrier, and we will calculate a *penetrability factor* to describe this reduction.

The second governing factor is the mechanism for trapping the particles permanently, once they have come inside the Coulomb barrier. One way is for a γ-ray to be emitted, and the particles lose that energy and fall down into a bound state of relative motion. This is a *direct capture* process, to be discussed further in the next section. Another direct mechanism is for one of the nuclei to be pushed up to an excited state, and energy absorbed this way that makes escape less likely.

Another way is for the particles to be captured by some of the long-living *resonances of the compound nucleus* that is formed of both the nuclei together. This will happen especially with heavier nuclei, where the is a high *level density* of these resonances: many per MeV. Compound nucleus resonances are usually narrow (that is, long-lived), so they capture flux from the direct reaction channels and release it later into other channels. Bohr's *independence hypothesis* is that the later release is independent of the details of the initial scattering state. These other channels may be neutron or γ emission, but these are called *statistical* or *evaporation* rather than direct processes. The direct processes are most likely with light nuclei, where the level density of resonances is rather low. The compound resonance in heavy nuclei can be simulated by an absorptive imaginary 'fusion part' to the optical potential, a part that is inside the Coulomb barrier.

2.2 Barrier Penetration

To calculate the *penetrability factors* for traversing a Coulomb barrier, we need to know the shape of the barrier, which is composed of a Coulomb repulsion at medium and large distances along with a nuclear attraction at short distances. We consider two cases in detail. In case (A) we neglect the nuclear potentials outside some radius R_m, and calculate just the penetrability factor for the Coulomb and centrifugal barriers. In case (B) we assume that the sum of all the potentials is like an upside down parabola around its maximum.

Once we have the penetrability factors $P_L(E)$ for each partial wave L, then a cross section will depend on them like

$$\sigma_X(E) = \frac{\pi}{k^2} \sum_L (2L+1) P_L(E) X_L \tag{38}$$

where X_L is a branching ratio to observing product X once the barrier has been passed.

(A) Penetrability factor for the Coulomb and centrifugal barriers: In this case, we only need to solve the Schrödinger equation with Coulomb and centrifugal potentials. Fortunately, the solutions in this case are well known as the *Coulomb functions* $F_L(\eta, kr)$ and $G_L(\eta, kr)$, and there are standard methods to calculate these. In terms of these functions, we normally define a *Coulomb* *penetrability factor*

$$P_L(R_m) = \frac{kR_m}{F_L(\eta, kR_m)^2 + G_L(\eta, kR_m)^2} \ , \tag{39}$$

where k is the asmptotic wave number, and the Sommerfeld parameter is $\eta = Z_1 Z_2 \mathrm{e}^2 \mu/(\hbar^2 k)$ for charges Z_1 and Z_2.

At low energies, the wave number k is small, and the denominator is dominated by G_L becoming very large as

$$F_L(\eta, \rho) \sim C_L(\eta)\rho^{L+1} \ , \ \ G_L(\eta, \rho) \sim \left[(2L{+}1)C_L(\eta)\ \rho^L\right]^{-1} \tag{40}$$

where $\rho = kR_m$,

$$C_0(\eta) = \sqrt{\frac{2\pi\eta}{\mathrm{e}^{2\pi\eta} - 1}} \ \ \text{and} \ \ C_L(\eta) = \frac{\sqrt{L^2 + \eta^2}}{L(2L{+}1)} C_{L-1}(\eta) \ . \tag{41}$$

This means that the s-wave (L=0) Coulomb penetrability factor tends to

$$P_0(R_m) \rightarrow kR_m \frac{2\pi\eta}{\mathrm{e}^{2\pi\eta} - 1} \sim kR_m 2\pi\eta \ \mathrm{e}^{-2\pi\eta} \tag{42}$$

as η gets large when k is small. This $\mathrm{e}^{-2\pi\eta}$ behaviour is the reason for the definition of the astrophysical S-factor $S(E)$ according to

$$\sigma(E) = \frac{\mathrm{e}^{-2\pi\eta}}{E} S(E) \ . \tag{43}$$

The $1/E$ factor here comes from the $1/k^2$ factor for the flux in an incoming plane wave.

(B) Penetrability factor for inverted parabolic barriers: Parabolic barriers are defined by their radial position R_B, their height E_B and their curvature from their second derivative $V''(R_B)$. The first derivative $V'(R_B) = 0$.

A normal harmonic oscillator potential is also parabolic, and from $V''(R_B) > 0$ we find the oscillator energy $\hbar\omega = \hbar\sqrt{V''(R_B)/\mu}$.

For our inverted barrier we define a similar 'characteristic quantum energy' $\hbar\omega = \hbar\sqrt{-V''(R_B)/\mu}$. This is used in the *Hill-Wheeler equation* (Phys. Rev. **89** (1953) 1102) for the penetrability:

$$P(E) = \frac{1}{1 + \mathrm{e}^{2\pi(E_B - E)/\hbar\omega}} \tag{44}$$

Note that just at the energy of the barrier top, with $E = E_B$, the penetrability is $P(E_B) = 0.5$.

This parabolic form for the barrier is most useful in heavy ion reactions, where there are barriers in many partial waves. If the barrier in each partial wave is at E_L with energy $\hbar\omega_L$, then the total fusion cross section is

$$\sigma_F(E) = \frac{\pi}{k^2} \sum_L \frac{2L{+}1}{1 + \mathrm{e}^{2\pi(E_L - E)/\hbar\omega_L}} \ . \tag{45}$$

since we are assume complete fusion once the barrier is passed ($X_L = 1$).

Wong (Phys. Rev. Lett. **31** (1973) 766) observed in practical cases that the ω_L tended to be constant, and that the radii of the barriers where also nearly constant. In that case the various partial wave barriers E_L may be simply given as

$$E_L = E_0 + \frac{L(L+1)\hbar^2}{2\mu R_0} \tag{46}$$

in terms of the s-wave barrier E_0 at radius R_0. If the sum in Eq. (45) is replaced by an integral, this integral may be done analytically to give

$$\sigma_F(E) = \frac{R_0^2 \hbar\omega_0}{2E} \ln\{1 + \exp[2\pi(E - E_0)/\hbar\omega_0]\} \tag{47}$$

For relatively large values of E this gives

$$\sigma_F(E) = \pi R_0^2 (1 - E_0/E) \tag{48}$$

and for relatively small values of $E \ll E_0$ we have

$$\sigma_F(E) = \frac{R_0^2 \hbar\omega_0}{2E} \exp[2\pi(E - E_0)/\hbar\omega_0] \ . \tag{49}$$

2.3 Compound Nucleus Decay

2.4 Multi-level Breit-Wigner Formula

The *multi-level Breit-Wigner formula* gives the total cross section as an incoherent sum of contributions from all the contributing resonances, each peaking near its position E_p with full width at half maximum of Γ_p. The compound-nucleus part of the angle-integrated cross section to channel α' from α is

$$\sigma_{\alpha'\alpha}(J^\pi_{\rm tot}; E) = \frac{\pi}{k^2}\, g_{J_{\rm tot}} \sum_p \frac{\Gamma_{\alpha p}\Gamma_{\alpha' p}}{(E-E_p)^2 + \Gamma_p^2/4} \ , \tag{50}$$

for resonance level p with spin and parity $J^\pi_{\rm tot}$, and total width of $\Gamma_p = \sum_\alpha \Gamma_{\alpha p}$. The spin-weighting factor is $g_{J_{\rm tot}} = (2J_{\rm tot}{+}1)/((2I_{p_i}{+}1)(2I_{t_i}{+}1))$ for initial spins I_{p_i} and I_{t_i}.

This expression will be most accurate if all the widths were much smaller than the mean spacing D between the levels, that is for mean widths $\langle\Gamma\rangle \ll D$. This is the case for non-overlapping and well separated resonances.

2.4.1 Hauser-Feshbach theory

If we add the cross sections of the multi-level Breit-Wigner formula (50) over all final states, the total should equal the summed reaction cross section σ_R for the loss of flux from the entrance channel. This should also match the total absorption cross section σ_A given by a one-channel optical potential for elastic scattering in channel α. Conversely, if we already know the optical potential and especially its imaginary component, we may use this knowledge to normalise the total widths for the decay of compound nuclear states. This is the basis for the very successful Hauser Feshbach approximation. for calculating the production and all the decay channels of compound nuclear states. We now derive the Hauser-Feshbach formula in the case of well separated resonances, using the above multi-level Breit-Wigner expression (50).

To find the cross sections averaged over many resonances, let us average over an interval I that contains many resonances: $I \gg D$ where D is the mean level spacing for CN resonances of given total spin and parity $J^\pi_{\rm tot}$. We define an *energy average cross section* for a sum of narrow peaks $\sigma_p(E)$ as appear in Eq. (50) by

$$\begin{aligned}
\langle\sigma(E)\rangle &= \frac{1}{I}\int_{E-I/2}^{E+I/2} \sum_p \sigma_p(E){\rm d}E \\
&= \frac{1}{I} \sum_{p,\ E_p\in[E\pm I/2]} \int_0^\infty \sigma_p(E){\rm d}E \\
&= \frac{1}{I}\frac{I}{D}\int_0^\infty \sigma_p(E){\rm d}E = \frac{1}{D}\int_0^\infty \sigma_p(E){\rm d}E
\end{aligned} \tag{51}$$

since there are I/D peaks within the averaging interval that are similar on average.

The integral over a single resonance of a Breit-Wigner resonance peak in Eq. (50) gives

$$\int_0^\infty {\rm d}E \frac{\Gamma_{\alpha p}\Gamma_{\alpha' p}}{(E-E_p)^2 + \Gamma_p^2/4} = \frac{2\pi\Gamma_{\alpha p}\Gamma_{\alpha' p}}{\Gamma_p} \tag{52}$$

so the energy average cross section is

$$\langle \sigma_{\alpha'\alpha}(J^{\pi}_{\rm tot}; E) \rangle = \frac{\pi}{k^2}\; g_{J_{\rm tot}} \left\langle \frac{\Gamma_{\alpha p}\Gamma_{\alpha' p}}{\Gamma_p} \right\rangle \frac{2\pi}{D}\ . \tag{53}$$

2.5 Width fluctuation corrections

Unfortunately the average *ratio* $\langle \Gamma_{\alpha p}\Gamma_{\alpha' p}/\Gamma_p \rangle$ is *not* simply given in terms of the average values of the numerator factors and the denominator. Because of possible correlations in the numerator averaging, we define a *width fluctuation factor* $W_{\alpha\alpha'}$ by

$$\left\langle \frac{\Gamma_{\alpha p}\Gamma_{\alpha' p}}{\Gamma_p} \right\rangle = W_{\alpha\alpha'} \frac{\langle \Gamma_\alpha \rangle \langle \Gamma_{\alpha'} \rangle}{\langle \Gamma \rangle}\ , \tag{54}$$

so the energy average cross section can be written

$$\langle \sigma_{\alpha'\alpha}(J^{\pi}_{\rm tot}; E) \rangle = \frac{\pi}{k_i^2}\; g_{J_{\rm tot}} W_{\alpha\alpha'} \frac{2\pi}{D} \frac{\langle \Gamma_\alpha \rangle \langle \Gamma_{\alpha'} \rangle}{\langle \Gamma \rangle}\ , \tag{55}$$

where the average sum $\langle \Gamma \rangle = \sum_\alpha \langle \Gamma_\alpha \rangle$.

To estimate the width fluctuation corrections $W_{\alpha\alpha'}$ in our $\Gamma \ll D$ limit, we focus on the numerators of Eq. (54). Factorising out the penetrability factors, the $W_{\alpha\alpha'}$ must satisfy $\langle \gamma^2_{p\alpha}\gamma^2_{p\alpha'} \rangle = W_{\alpha\alpha'} \langle \gamma^2_{p\alpha} \rangle \langle \gamma^2_{p\alpha'} \rangle$. For inelastic reactions $\alpha \neq \alpha'$, so the $\gamma^2_{p\alpha}$ and $\gamma^2_{p\alpha'}$ should be statistically independent, giving $\langle \gamma^2_{p\alpha}\gamma^2_{p\alpha'} \rangle \sim \langle \gamma^2_{p\alpha} \rangle \langle \gamma^2_{p\alpha'} \rangle$ and $W_{\alpha\alpha'} \approx 1$. For elastic channels with $\alpha = \alpha'$, however, we have $\langle \gamma^4_{p\alpha} \rangle = W_{\alpha\alpha} \langle \gamma^2_{p\alpha} \rangle^2$, so, defining $x = \gamma_{p\alpha}/\langle \gamma^2_{p\alpha} \rangle^{1/2}$, the elastic $W_{\alpha\alpha}$ is the fourth moment $\langle x^4 \rangle$. If the $\gamma_{p\alpha}$ follow the Porter-Thomas distribution, then this is the fourth moment of a normal distribution, namely $W_{\alpha\alpha} = 3$. To conserve the total cross section, however, this enhancement of the elastic channel should be compensated by a proportional reduction of all the other channels. This will only be significant if there are not too many of those.

For incoming channel α, the total reaction cross section is the sum of all these terms over the outgoing channels α', and since again $\sum_{\alpha'} \langle \Gamma_{\alpha'} \rangle = \langle \Gamma \rangle$ we have

$$\begin{aligned} \langle \sigma^R_\alpha(J^{\pi}_{\rm tot}; E) \rangle &= \sum_{\alpha'} \langle \sigma_{\alpha'\alpha}(J^{\pi}_{\rm tot}; E) \rangle \\ &= \frac{\pi}{k^2} g_{J_{\rm tot}} \frac{2\pi \langle \Gamma_\alpha \rangle}{D} \end{aligned} \tag{56}$$

where we have set $W_{\alpha\alpha'} \approx 1$ since we are summing over the non-elastic channels.

2.6 Transmission coefficients

We now establish the absolute scale of the $\langle \Gamma_\alpha \rangle$ by connection with the reaction cross sections predicted by the optical model. The optical model gives a total reaction cross section of

$$\sigma^R_\alpha(J^{\pi}_{\rm tot}; E) = \frac{\pi}{k^2} g_{J_{\rm tot}} (1 - |\mathbf{S}^{J_{\rm tot}\pi,{\rm opt}}_{\alpha\alpha}|^2)\ , \tag{57}$$

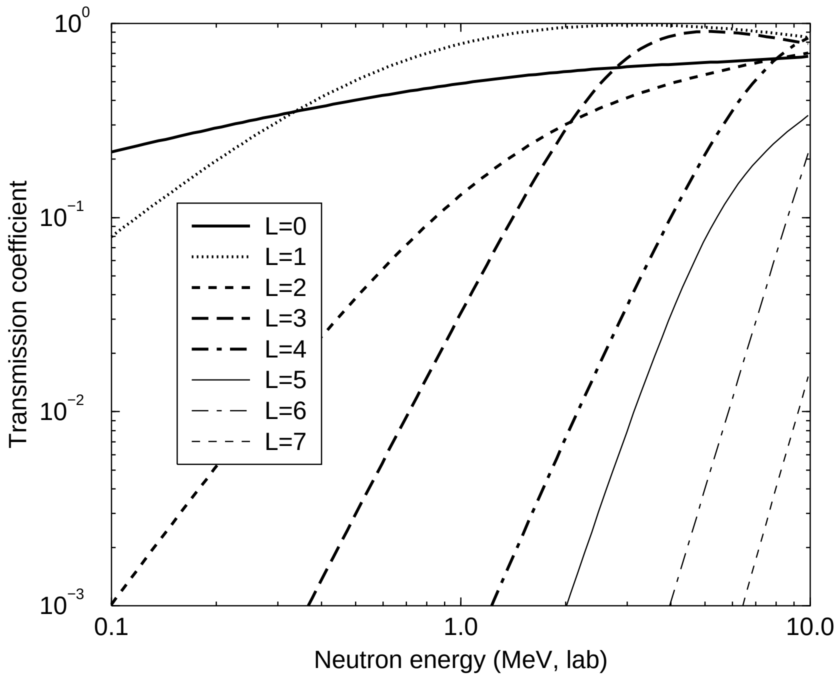

Figure 3: Transmission coefficients for neutrons incident on ^{90}Zr in various partial waves L, using the global optical potential of Koning and Delaroche.

where $\mathbf{S}^{\rm opt}_{\alpha\alpha}$ is the elastic optical S-matrix element, and comparison of these two expressions gives

$$1 - |\mathbf{S}^{\rm opt}_{\alpha\alpha}|^2 = \frac{2\pi\langle\Gamma_\alpha\rangle}{D} \ . \tag{58}$$

We now define the so-called *transmission coefficients*[1]:

$$\mathcal{T}_\alpha = 1 - |\mathbf{S}^{\rm opt}_{\alpha\alpha}|^2 \ , \tag{59}$$

which measure the 'coupling' described by the imaginary potentials between the external scattering and the internal compound-nucleus production. Fig. 3 shows these for neutrons on ^{90}Zr in various partial waves.

In terms of the transmission coefficients, Eq. (55) becomes

$$\langle\sigma_{\alpha'\alpha}(J^\pi_{\rm tot}; E)\rangle = \frac{\pi}{k^2}\ g_{J_{\rm tot}} W_{\alpha\alpha'} \frac{\mathcal{T}_\alpha \mathcal{T}_{\alpha'}}{\sum_{\alpha''}\mathcal{T}_{\alpha''}} \ . \tag{60}$$

We see in this expression what are called the *Hauser-Feshbach branching ratios*:

$$\mathcal{B}_{\alpha'} = \frac{\mathcal{T}_{\alpha'}}{\sum_{\alpha''}\mathcal{T}_{\alpha''}} \ . \tag{61}$$

These Hauser-Feshbach formulae have simple physical interpretations. The branching ratio for the decay $\mathcal{B}_{\alpha'}$ of a given compound nuclear state to final channel α' is proportional to the transmission

[1] They should be distinguished from e.g. the previous barrier tunnelling coefficients $P(E)$, because these $\mathcal{T}_\alpha = 0$ for all real optical potentials, and $\mathcal{T}_\alpha = 1$ for strongly absorbed partial waves.

coefficient $\mathcal{T}_{\alpha'}$, normalised in the denominator by the summed coefficient so the total decay probability is unity. The *production* of the CN state is assumed to be a time-reversal of the decay mechanism, and hence is proportional to the same $\mathcal{T}_\alpha$.

The Hauser-Feshbach formula to calculate all the decay chains that occur after compound nucleus production, if we know independently the average level spacings D.

2.7 Weisskopf-Ewing approximation

The cross section summed over all total spins and parities is

$$\langle \sigma_{\alpha'\alpha}(E) \rangle = \frac{\pi}{k^2} \sum_{J^\pi_{\rm tot}} g_{J_{\rm tot}} W_{\alpha\alpha'} \frac{\mathcal{T}_\alpha \mathcal{T}_{\alpha'}}{\sum_{\alpha''} \mathcal{T}_{\alpha''}} \ . \tag{62}$$

This expression does *not* factorise into a product of production and decay probabilities, because $J_{\rm tot}$ and parity π are conserved quantum numbers that are not affected by averaging. It only factorises if $W_{\alpha\alpha} \approx 1$ *and* the $J^\pi_{\rm tot}$ sum can be ignored. This latter occurs for example if only one incoming partial wave α is significant (e.g. $1/2^+$ for thermal neutrons on a light spin-zero target), or if all the $\mathcal{T}_\alpha$ have the same $J^\pi_{\rm tot}$ dependence. The earlier Weisskopf-Ewing theory can be obtained as a limit of the Hauser-Feshbach theory by assuming a fixed distribution of spins. This enables a complete factorisation as

$$\langle \sigma_{\alpha'\alpha}(E) \rangle = \sigma^R_\alpha(E) \frac{\mathcal{T}_{\alpha'}}{\sum_{\alpha''} \mathcal{T}_{\alpha''}} \equiv \sigma^R_\alpha(E)\, \mathcal{B}_{\alpha'} \ . \tag{63}$$

This theory simply states that the total reaction cross $\sigma^R_\alpha(E)$ of Eq. (57) decays according to branching ratios $\mathcal{B}_{\alpha'}$ obtained by normalising the $\mathcal{T}_{\alpha'}$ to unit probability.