Q3. $$\int_0^\infty dr\, u_\ell^*(k,r)\, u_\ell(k',r) = \frac{\pi}{2}\delta(k-k').$$

$$\hat{u}_i = C_i \int_{\Delta k_i} g(k)\, u_\ell(k,r)\, dk.$$

$$\int_0^\infty dr\, \hat{u}_i^*(r)\, \hat{u}_j(r) = C_i^* C_j \int_{\Delta k_i} dk \int_{\Delta k_j} dk'\, g^*(k)\, g(k')$$

$$\left.\begin{array}{l} k \text{ in } \Delta k_i \\ k' \text{ in } \Delta k_j \end{array}\right\} \qquad \times \underbrace{\int_0^\infty u_\ell^*(k,r)\, u_\ell(k',r)\, dr}_{\frac{\pi}{2}\delta(k-k')}$$

if $\Delta k_i \neq \Delta k_j$, $i \neq j$, $\delta(k-k') = 0$. $\forall\, k, k'$.

Normalisation, $i = j$, $\Delta k_i = \Delta k_j$

$$\int_0^\infty dr\, \hat{u}_i^*(r)\, \hat{u}_i(r) = |C_i|^2 \frac{\pi}{2} \int_{\Delta k_i} dk \int_{\Delta k_i} dk' \times |g^*(k)\, g(k')\, \delta(k-k')$$

$$= \frac{\pi}{2} |C_i|^2 \underbrace{\int_{\Delta k_j} dk\, |g(k)|^2}_{\text{real.}} = 1 \text{ for normalised wave functions}$$

$$|C_i|^2 = \frac{2}{\pi N_i}, \quad C_i = \sqrt{\frac{2}{\pi N_i}}, \quad N_i = \int_{\Delta k_i} dk\, |g(k)|^2$$