**TALENT Course 6: Theory for exploring nuclear reaction experiments**
**Exercises: Transfer reactions**
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## Analytical/Mathematical exercises

1. **Finite-range effects: Background:** We consider the ${}^{A}_{Z}X(gs)(d,p){}^{A+1}_{Z}Y(j^{\pi})$ and ${}^{A+1}_{Z}Y(gs)(p,d){}^{A}_{Z}X(j^{\pi})$ single-neutron transfer reactions to be definite. For such reactions (think e.g. about the $(p,d)$ pickup process) the short ranged neutron-proton interaction, $V_{np}(r)$, is the interaction responsible for the transfer reaction. $V_{np}(r)$ enters the QM transition amplitude $\langle f|V_{np}|i\rangle$ through the product of $V_{np}(r)$ with the ground-state wave function of the deuteron $\phi_d(r)$. Due to $V_{np}(r)$, this product, $D(r) = V_{np}(r)\phi_d(r)$, often referred to as the vertex function, has a relatively short but finite range in the neutron-proton separation $r$.

   To a very good approximation $\phi_d(r)$ is an $s$-state wave function ($\approx 95\%$), so

$$\phi_d(r) = [u_0(r)/r]\, Y_{00}(\hat{\mathbf{r}})$$

   where $u_0(r)$ satisfies the radial Schrödinger equation

$$\left[-\frac{\hbar^2}{2\mu}\frac{d^2}{dr^2} + V_{np}(r)\right] u_0(r) = -\varepsilon_d u_0(r) \ ,$$

   with binding energy $\varepsilon_d = \hbar^2\alpha^2/2\mu = 2.226$ MeV. $Y_{00}(\hat{\mathbf{r}})$ is the $s$-state spherical harmonic and has value $1/\sqrt{4\pi}$.

   The leading finite-range effect in $D(r)$ can be quantified in momentum space, via the Fourier transform of the vertex function. For the wave function above

$$D(k) = \int d\mathbf{r}\, \exp(-i\mathbf{k}\cdot\mathbf{r}) V_{np}(r)\phi_d(r) = \sqrt{4\pi}\int dr\, r j_0(kr) V_{np}(r) u_0(r) \ . \quad (1)$$

   Making the *zero-range (ZR) approximation*, that $D(r) = V_{np}(r)\phi_d(r) \approx D_0\delta(\mathbf{r})$, then clearly $D(k) \to D_0$, a constant, and

$$D_{ZR}(k) \approx D_0 = \int d\mathbf{r}\, V_{np}(r)\phi_d(r) = \sqrt{4\pi}\int dr\, r V_{np}(r) u_0(r) \ . \quad (2)$$

   So $D_0$ determines the overall strength of the transfer interaction. The leading order *finite-range* correction to $D(k)$ is included by keeping the first term in the expansion of the vertex $D(k)$ for finite $k$, making use of the expansion of the spherical Bessel function (recall that $j_0(kr) = \sin kr/(kr)$), i.e.

$$j_0(kr) = 1 - k^2r^2/6 + \ldots$$

in equation (1), so that

$$D_{FR}(k) \approx \sqrt{4\pi} \int dr r \left[1 - \frac{k^2 r^2}{6}\right] V_{np}(r) u_0(r) = D_0[1 - k^2 \mathcal{R}^2]. \qquad (3)$$

Here $\mathcal{R}$ (with units of length) is the *finite-range length parameter*.

**Exercise:** A simple (but realistic) model of the deuteron wave function is the Hulthén wave function, where

$$u_0(r) = N_0 \left[e^{-\alpha r} - e^{-\beta r}\right], \qquad N_0 = \sqrt{\frac{2\alpha\beta(\alpha+\beta)}{(\beta-\alpha)^2}}, \qquad \beta = 6\alpha \ .$$

Obtain formulae for and compute numerical values for the finite-range constants $D_0$ (MeV fm$^{3/2}$) and $\mathcal{R}$ (fm) for this Hulthén wave function. You can use the Schrödinger equation to deduce the product $V_{np}(r)u_0(r)$.

2. **Spectroscopic factors: Background:** In the lectures on nuclear structure considerations, the spectroscopic factors (SF) $\mathcal{S}(\ell j)$ for one-nucleon *removal* reactions were discussed. In particular, we showed that in the independent particle model (IPM), and for the removal of the nucleon from a sub-shell $j$ containing $n$ identical nucleons, that

$$\mathcal{S}(\ell j) = n \left((j^{n-1})b, j; a|\right\}(j^n)\, a)^2 \ .$$

We will now denote these *nucleon removal* reaction SF more precisely, and we use $\mathcal{S}^-(\ell j; a, n \to b, n-1)$. We do this because we need to also discuss SF for *nucleon adding* reactions, that will determine the yields (cross sections) in reactions such as the $(d,p)$ or $(^3He, d)$ transfer processes. For such reactions we will denote the SF as $\mathcal{S}^+(\ell j; b, n-1 \to a, n)$.

We also learned that the $\mathcal{S}^-$ satisfied the sum rule

$$\sum_b \mathcal{S}^-(\ell j; a, n \to b, n-1) = n \sum_b ((j^{n-1})b, j; a|\}(j^n)\, a)^2 = n \ ,$$

the number of particles in the $j$-orbital from which the nucleon is being removed in the heavier of the nuclei.

The $\mathcal{S}^+$ are defined such that

$$\mathcal{S}^+(\ell j; b, n-1 \to a, n) = \left\{\frac{2a+1}{2b+1}\right\} \mathcal{S}^-(\ell j; a, n \to b, n-1), \qquad (4)$$

from which it follows that the sum rule for the SF to all final states is

$$\sum_a \mathcal{S}^+(\ell j; b, n-1 \to a, n) = (2j+1) - (n-1) \ ,$$

and equals the number of vacancies or *holes* in the $j$-orbital to which the nucleon is being added in the lighter nucleus.

**Exercise:** Revisit your earlier solutions to Problem 1 of the overlaps exercises of Tuesday of Week 1 and calculate the SF, $\mathcal{S}^+$, for adding a neutron to the $\nu 1f_{7/2}$ orbital for the $^{47,46,\ldots 41,40}$Ca systems based on the independent particle model. Show in at least one odd-$n$ and one even-$n$ case that the $\mathcal{S}^+$ sum rule of Eq. (4) is also satisfied.

**Note**: The importance of this formula is that it shows that, if the nuclear structure overlap was the *only consideration*, the nucleon adding and nucleon removing SF's (and their cross sections for transitions between the two states $a\alpha \leftrightarrow b\beta$) would differ by the factor $(2a+1)/(2b+1)$. Transfer reaction codes include this factor automatically in the spin algebra and so, when calculated cross sections are multiplied by a theoretical SF, or fitted to deduce an empirical SF, the SF used is $\mathcal{S}^-(\ell j; a, n \to b, n-1)$.

**cfps**: The relevant coefficients of fractional parentage are again written as

$$((j^{n-1})b, j; a|\}(j^n)\, a) \equiv ((j^{n-1})\, v_f\, b, j; a|\}(j^n)\, v_i\, a)$$

and for the low seniority states, shown as $(v_f \leftrightarrow v_i)$ below, are:

for $n$ even

$$((j^{n-1})j, j; 0|\}(j^n)\, 0) = 1,\ (1 \leftrightarrow 0),$$

$$((j^{n-1})j, j; J|\}(j^n)\, J) = \left(\frac{2(2j+1-n)}{n(2j-1)}\right)^{\frac{1}{2}},\ (1 \leftrightarrow 2),\ J \neq 0 \text{ and even},$$

and for $n$ odd

$$((j^{n-1})0, j; j|\}(j^n)\, j) = \left(\frac{2j+2-n}{n(2j+1)}\right)^{\frac{1}{2}},\ (0 \leftrightarrow 1),$$

$$((j^{n-1})J, j; j|\}(j^n)\, j) = -\left(\frac{2(n-1)(2J+1)}{n(2j-1)(2j+1)}\right)^{\frac{1}{2}},\ (2 \leftrightarrow 1),\ J \neq 0 \text{ and even}.$$

http://www.nucleartheory.net/NPG/Talent_material/TALENT_lectures/cfps.pdf

### Computational reactions calculations

The programs `front_twofnr` and `brush` are useful in interactively assembling transfer reaction data sets for `twofnr` and `fresco`. These prompt for and so make one aware of all of the inputs that need to be specified to carry out such calculations.
In such calculations, the Johnson-Soper adiabatic approximation is an approximate but very efficient way to deal with the effects of breakup on a transfer reaction for the cases of an $s$-state projectile wave function (such as for the deuteron and (d,p) and/or (p,d) nucleon transfer reactions). An excellent short account of the inputs requested by `front_twofnr` and `brush` can also be found at the link named `Glendenning`.

1. (a) Use `front_twofnr` to construct data sets `tran.xxx` (or `brush` as an alternative data set generator) for zero-range DWBA and zero-range Johnson-Soper adiabatic model calculations for the $^{12}$C$(d,p)^{13}$C reactions at 25 MeV deuteron incident energy – that populate the $1/2^-$ (ground state), $1/2^+$ (3.09 MeV), $3/2^-$ (3.68 MeV) and $5/2^+$ (3.85 MeV) final states of $^{13}$C. The experimental $^{13}$C spectrum is at the link `13C levels spectrum`. You should use reasonable choices of global optical potentials or use the JLM G-matrix interaction. For the JLM case you will have to pre-calculate the target and residual nucleus densities using `dens`.

   (b) Perform the transfer reaction calculations using `twofnr` (or `fresco`) and compare the quality of agreement of both the DWBA and the Johnson-Soper calculations with the experimental data of Toyokawa *et al.* (Thesis, RCNP/Osaka University 1995). These are given as a function of the angle of the outgoing proton in the c.m. frame in the files `12Cdp12-25.exp`, `12Cdp12+25.exp`, `12Cdp32-25.exp` and `12Cdp52+25.exp`. The cross section calculated by `twofnr` is written to the file `21.xxx`. The experimental $^{13}$C spectrum is at the link `13C levels spectrum`.

   (c) The leading finite-range correction to the zero-range approximation was studied in the analytical exercise of this exercises sheet. How do the calculated cross sections change if you include the (default) approximate treatment of the finite-range of the nucleon-nucleon interaction? (If time is short, try this for at least the ground-state case.)

   (d) Revisit the shell model tutorial from the Tuesday exercises of Week 1 of this Talent course. Using the PJT interaction in the $p$-shell model space, calculate the shell-model spectroscopic factors for these $^{12}$C(g.s.)$(d,p)^{13}$C$(J^\pi)$ reactions to the $1/2^-$ and $3/2^-$ final states of $^{13}$C. Compare these with the approximate spectroscopic factors for each final state that you obtain by scaling your theoretical curves (that use spectroscopic factors $\mathcal{S} = 1$, called the unit cross sections) to the data. Are there any surprises here?

   (e) Repeat the calculations for $^{12}$C$(d,p)^{13}$C$(1/2^-$,g.s.) at $E_d = 10$ MeV and extract the spectroscopic factor (as a ratio of the experimental to the theoretical cross sections at forward angles) and the corresponding asymptotic normalization coefficient. The experimental data at 10 MeV can be found at
   `https://groups.nscl.msu.edu/nscl_library/pddp/database.html`,
   linked at the web-site. Compare with the results in (d) and draw your conclusions.

   (f) Study the dependence of the spectroscopic factors and asymptotic normalization coefficients on the details of the bound state effective interaction (for example if the radius is changed).