TALENT Course 6: Theory for exploring nuclear reaction experiments
Exercises: High energy - Eikonal Methods
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Analytical/Mathematical exercises

1. **Eikonal solution**:

   The eikonal approach replaces the scattering wave function solution $\psi_{\mathbf{k}}^{+}(\mathbf{r})$ of the two-body Schrödinger equation for a *core* and a *valence* particle, i.e.

$$\left(-\frac{\hbar^2}{2\mu}\nabla_r^2 + U(r) - E_{cm}\right)\psi_{\mathbf{k}}^{+}(\mathbf{r}) = 0, \qquad \mu = \frac{m_c m_v}{m_c + m_v},$$

   by the product form

$$\psi_{\mathbf{k}}^{(+)}(\mathbf{r}) = \exp(i\mathbf{k}\cdot\mathbf{r})\omega(\mathbf{r}).$$

   Here $\omega(\mathbf{r})$ is a modulating function that modifies the scattering wave from its incident plane wave form and $\omega(\mathbf{r})$ includes all the effects due to the potential $U(r)$. If particles are incident in the $\hat{\mathbf{z}}$ direction, $\mathbf{k} = k\hat{\mathbf{z}}$, then the scattering boundary condition requires that $\omega(\mathbf{r}) \to 1$ as $\mathbf{r} \to \infty$ in the $-\hat{\mathbf{z}}$ direction, i.e. $z \to -\infty$.

   (i) Substitute the assumed product form $\psi^{(+)} = e^{i\mathbf{k}\cdot\mathbf{r}}\omega$ in the Schrödinger equation and deduce the differential equation satisfied by the function $\omega(\mathbf{r})$.
   (ii) Assuming that the (curvature) term, $\nabla_r^2\omega$, can be neglected show that the modulation function has solution

$$\omega(\mathbf{r}) = \exp\left[-\frac{i\mu}{\hbar^2 k}\int_{-\infty}^{z} U(r')\,dz'\right] = \exp\left[-\frac{i}{\hbar v}\int_{-\infty}^{z} U(r')\,dz'\right].$$

2. **Eikonal phase shifts**:

   Evaluate analytically the form of the eikonal phase shift functions $\chi(b)$ for impact parameters b and hence the form of the eikonal S-matrices, $S(b)$, where

$$S(b) = \exp\left[i\chi(b)\right] = \exp\left[-\frac{i}{\hbar v}\int_{-\infty}^{\infty} U(r')\,dz'\right]$$

   in the cases that the potential energy $U(r)$ is:

   (a) complex and constant, $U(r) = -(V_0 + iW_0)$, for $r < R$ and $U(r) = 0, r > R$,

   (b) complex and Gaussian, with range R, i.e. $U(r) = -(V_0 + iW_0)\exp(-r^2/R^2)$.

3. **Absorptive and removal cross sections**:

   A fast, bound, two-body projectile is composed of particles 1 and 2 and interacts with a target nucleus. If the eikonal S-matrices for the scattering of particles 1 and 2 with the target are $S_1(b_1)$ and $S_2(b_2)$, the absorption cross section in the scattering of the composite projectile is determined by the operator

$$\mathcal{A}_{12} = (1 - |S_1 S_2|^2) = (1 - |S_1|^2 |S_2|^2).$$

   Show that

$$\mathcal{A}_{12} = (1 - |S_1|^2)|S_2|^2 + |S_1|^2(1 - |S_2|^2) + (1 - |S_1|^2)(1 - |S_2|^2)$$

   and interpret each term if 1 labels a nucleon and 2 labels a heavy projectile core nucleus. Generalise this expression for 3 bodies and $\mathcal{A}_{123}$. Identify which terms need to be computed to describe two-nucleon removal reactions if we assume that 1 and 2 label two nucleons and 3 labels the heavy projectile residue (core) nucleus.

Computational exercises

Elastic scattering of nucleons, ions and composites

For the following you will need to run `FRESCO`, `eikonal_s`, `glauber` and `knockout`. Outlines for `eikonal_s`, `knockout` and `glauber` and sample data sets for `eikonal_s` and `glauber` can be found on the web-site.

1. Calculate the eikonal S-matrix as a function of the impact parameter, $|S(b)|$, for a neutron scattering from a ^{12}C target at 50 and 100 MeV per nucleon. You should use the complex optical potential with real part (V, r, a) = (37.4 MeV, 1.2 fm, 0.75 fm) and a volume-shaped absorptive (imaginary) term with (W_v, r_v, a_v) = (10.0 MeV, 1.3 fm, 0.60 fm). You should use the program `eikonal_s`. Calculate the elastic scattering differential cross section angular distributions at the two energies, both exactly (using `FRESCO`) and using the eikonal approximation. For the eikonal calculation you can use the program `glauber` that will take the eikonal $S(b)$ as an input. You can also copy a `FRESCO` n+^{12}C data set from the web-site. You should be familiar with elastic scattering calculations using `FRESCO`.

2. Calculate the eikonal S-matrix as a function of the impact parameter for ^{10}Be elastic scattering from a ^{12}C target, also at 50 and 100 MeV per nucleon. You should use the complex potential with real part (V, r, a) =(123.0 MeV, 0.75 fm, 0.8 fm) and volume-shaped absorptive (imaginary) term with (W_v, r_v, a_v) =(65.0 MeV, 0.78 fm, 0.8 fm) and a Coulomb radius parameter of 1.2 fm. You should use the program `eikonal_s`.

   Calculate the elastic scattering differential cross section angular distributions for the two energies, both exactly (using `FRESCO`) and using the eikonal approximation using the program `glauber`. In this nucleus plus nucleus scattering case all radius parameters are multiplied by $10^{1/3} + 12^{1/3}$. You can copy a `FRESCO` data set for ^{10}Be+^{12}C elastic scattering from the web-site.

3. Use
(a) the S-matrices that you calculated above, $S_c(b_c)$ and $S_n(b_n)$ for the ^{10}Be- and neutron-^{12}C systems at 50 MeV/nucleon (or recalculate them if needed), and
(b) a bound state wave function ϕ_0 for the ground state of ^{11}Be, to calculate the elastic scattering of the halo nucleus ^{11}Be from ^{12}C at 50 MeV/nucleon [see e.g. the link `PHYSICAL REVIEW C55,R1018` at web-site].

   You should use the `output S-matrix?` option in `knockout` to calculate the elastic S-matrix of the ^{11}Be composite projectile with the target, i.e. $S_{11}(b) = \langle\phi_0|S_c(b_c)S_n(b_n)|\phi_0\rangle$. This can then be used with `glauber` to calculate the elastic cross section angular distribution. The experimental data shown in the PRC Paper (ratio to Rutherford) are also available at the web-site. Compare the elastic cross sections for ^{11}Be and ^{10}Be.

Knockout cross sections and momentum distributions

For the following you will need to run `dfold_front`, `dfold`, `knockout` and `momentum`, etc. Outlines and/or sample data sets can be found on the course web-site.

1. Reproduce the cross sections and momentum distributions of ^{22}O after neutron removal reactions from ^{23}O at 75 MeV per nucleon on a carbon target. Use the separation energies and the default bound states potential geometries shown in the lecture slides and use single and double folding model interactions for the ^{22}O- and neutron-carbon systems. These will require the ^{22}O (Hartree-Fock) and ^{12}C (Gaussian) densities.

2. Calculate the neutron-removal reactions from ^{11}Be and ^{15}C on a ^{9}Be target at 60 and 54 MeV per nucleon, respectively. Consider these projectiles as weakly-bound ^{10}Be + neutron and ^{14}C + neutron two-body systems. We will make use of the eikonal approximation (Lectures) and so we will need S-matrices for
(i) the neutron-^{9}Be system at 54 and 60 MeV,
(ii) the ^{10}Be + ^{9}Be system at 60 MeV/nucleon, and
(iii) the ^{14}C + ^{9}Be system at 54 MeV/nucleon.
(a) calculate the Hartree Fock densities of the ^{14}C and ^{10}Be core nuclei, using `dens`,
(b) assuming the matter density of the ^{9}Be target is a Woods-Saxon (Fermi) shape with an rms matter radius of 2.5 fm and diffuseness of 0.5fm, calculate the required S-matrices for ^{10}Be + ^{9}Be and ^{14}C + ^{9}Be using `front_dfold`, `dfold`, and `eikonal_s`
(c) calculate the neutron + ^{9}Be S-matrices from the JLM nucleon optical potential, using `jlm` and `eikonal_s`, also at 54 and 60 MeV.

3. (a) Estimate the cross sections for knockout and their momentum distributions (using `knockout` and `momentum`) for the single neutron knockout reaction from ^{15}C at 54 MeV per nucleon. You should perform calculations for the 0^+(ground state) and 1^- (6.09 MeV) final state transitions: earlier results for these are shown at the link `Tostevin_15C` (web-site). The experimental momentum distributions can be

found in the files `c15gs.exp.dat` and `c15ex.exp.dat` for the ground and excited states, respectively, at the web-site.
(b) Calculate the momentum distributions for the 10Be residues following knockout of a neutron from a 11Be beam at 60 MeV/nucleon, based on assuming that the neutron was bound in an s-, p- or d-state with separation energy of 0.5 MeV, as are shown in Figure 2 at the link `Aumann paper` (web-site).