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REDUCTION OF THE THREE-PARTICLE COLLISION PROBLEM TO MULTI-CHANNEL TWO-PARTICLE LIPPMANN-SCHWINGER EQUATIONS

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Abstract: Starting from Faddeev's theory, we derive exact one-dimensional integral equations for the three-body scattering problem. The task is accomplished by application of the Schmidt method (quasi-particle method). Our equations, which are more practical than those of Faddeev, have the structure of multi-channel two-particle Lippmann-Schwinger equations. The generalized "potentials" occurring in them have to be calculated from two-dimensional integral equations, the iteration of which should rapidly converge in most cases of physical interest. Thus our approach admits the systematical application of the methods of perturbation theory to composite-particle scattering problems. To lowest order, we reproduce a result of Lovelace.

1. INTRODUCTION

Collision processes of three non-relativistic particles which may form bound two-particle subsystems, have been studied with increasing interest during the last years. The problem, of course, can be formulated in a well-defined manner by the basic concepts of multi-channel scattering theory. However, to handle it in practice, manageable integral equations ought to be formulated for scattering states or transition operators. A decisive step towards this goal has been accomplished by Faddeev [1, 2]. The main feature of his equations is that their kernel becomes a Hilbert-Schmidt operator after iteration. Thus, in principle being solvable, it appears however that they are too complicated for practical purposes. In particular, these equations are still two-dimensional after angular momentum decomposition. To obtain a more practical theory, it therefore seems desirable to derive *exact*, one-dimensional integral equations for the transition amplitudes.

Starting from Faddeev-like equations for appropriately chosen transition operators, we reduce them to one-dimensional integral equations which have the form of multi-channel two-particle Lippmann-Schwinger equations. This is done by application of the Schmidt method already used by Weinberg for the treatment of the two-particle collision problem *. We start by split-

* In this form of application, it is called "quasi-particle method", see refs. [3-5].

ting the transition operators of the two-particle subsystems occurring in the Faddeev equations into separable and non-separable parts. It should be stressed that we do not use the fact that the square of the Faddeev kernel, being a Hilbert-Schmidt operator, may be approximated by a series of terms separable with respect to all of its variables; i.e. this main property of the Faddeev equations, which is generally assumed to be essential for the applicability of the Schmidt method [6], is not important for our treatment *. (This point will be discussed in more detail in a subsequent publication.) As a consequence of the fact that our separable parts are separable only with respect to the subsystem variables, we are not left with purely algebraic equations as in the two-particle case but are left with one-dimensional integral equations for the three-particle transition amplitudes.

The generalized "potentials" occurring in these equations are to be determined from Faddeev-type integral equations, the kernel of which only involves the non-separable parts of the two-particle subsystem transition amplitudes. Since we can make these non-separable parts as small as we like **, we may solve these equations by iteration.

In general it may suffice to extract only those separable parts of the two-particle transition amplitudes which correspond to their bound-state and resonance poles. We then reproduce in lowest order of iteration the Lovelace equations [8]. A test of the validity of these approximative equations becomes therefore possible.

We so derive an exact scheme to solve the three-particle problem which is also practical because we first reduce the two-dimensional Faddeev equations to one-dimensional Lippmann-Schwinger-type equations, the "potentials" of which can be determined by a rapidly converging perturbation series. Or, stated in more general terms, we show in the case of three-particle scattering that the strong interaction problem is amenable, after suitably extracting two-particle bound states and resonances, to methods of perturbation theory.

In sect. 2, we briefly recall some basic procedures of formal scattering theory and introduce transition operators which are of a rather symmetric structure. Therefore they are more convenient for the following considerations than the transition operators generally used in the literature [7, 8]. Furthermore we give the Faddeev-like equations which they obey. Sect. 3 is concerned with the transformation of these equations to multi-channel Lippmann-Schwinger-type equations. In sect. 4, we consider the three-particle bound-state problem. Finally in sect. 5, two special cases of physical importance are discussed as examples where our equations simplify considerably.

* For our procedure it is only a matter of convenience to start from Faddeev equations. Especially it is useful, but not decisive, that these contain explicitly the transition operators of the subsystems instead of the potentials.

** This is due to the fact that the kernel of the two-particle Lippmann-Schwinger equation is a Hilbert-Schmidt operator [7].

2. TRANSITION OPERATORS AND FADDEEV EQUATIONS

In the following we consider three distinguishable particles * with two-particle interactions

$$V_\alpha = V_{ij} , \qquad i, j = 1, 2, 3 \; ; \; \alpha \neq i, j , \qquad V_0 \equiv 0 . \tag{2.1}$$

Besides the total Hamiltonian

$$H = H_0 + \sum_{\alpha=1}^{3} V_\alpha = H_0 + V , \tag{2.2}$$

we introduce channel Hamiltonians by

$$H_\alpha = H_0 + V_\alpha , \tag{2.3}$$

the eigenstates of which are called $|\Phi_{\alpha, m}\rangle$

$$H_\alpha |\Phi_{\alpha, m}\rangle = E_{\alpha m} |\Phi_{\alpha, m}\rangle . \tag{2.4}$$

From time-dependent scattering theory, we derive as usual ** the representation of the S-matrix for the transition from an initial configuration characterized by the state *** $|\Phi_{\alpha, m}\rangle$ to a final configuration $|\Phi_{\beta, n}\rangle$

$$S_{\beta n;\alpha m} = \lim_{t\to\infty} \lim_{\epsilon\to 0} e^{i(E_{\beta n} - E_{\alpha m})t} \langle \Phi_{\beta,n} | (-i\epsilon)\, G(E_{\alpha m} + i\epsilon) | \Phi_{\alpha,m} \rangle . \tag{2.5}$$

Here we have introduced the resolvent

$$G(z) = (H - z)^{-1} , \tag{2.6}$$

which satisfies the (second) resolvent equations

$$G(z) = G_\alpha(z) - G_\alpha(z) \overline{V}_\alpha G(z) , \tag{2.7}$$

$$G(z) = G_\alpha(z) - G(z) \overline{V}_\alpha G_\alpha(z) , \tag{2.8}$$

with

$$G_\alpha(z) = (H_\alpha - z)^{-1} , \tag{2.9}$$

$$\overline{V}_\alpha = V - V_\alpha = \sum_{\gamma \neq \alpha} V_\gamma . \tag{2.10}$$

Inserting now (2.8) in (2.7), we get

* For the treatment of identical particles and the inclusion of the Pauli principle, see ref. [8].

** For the basic concepts of multi-channel scattering theory, see e.g., ref. [9] .

*** In the following we always assume $\alpha \neq 0$. We therefore do not consider the case of three incoming free particles.

$$G(z) = \delta_{\beta\alpha} G_\beta(z) - G_\beta(z) U_{\beta\alpha}(z) G_\alpha(z) , \tag{2.11}$$

where

$$U_{\beta\alpha}(z) = -(1 - \delta_{\beta\alpha})(H_0 - z) + V - V_\alpha - V_\beta + \delta_{\beta\alpha} V_\alpha - \bar{V}_\beta G(z) \bar{V}_\alpha . \tag{2.12}$$

The representation (2.11) of the resolvent $G(z)$ allows us to perform the limits in (2.5). The result

$$S_{\beta n;\, \alpha m} = \delta_{\beta\alpha}\delta_{nm} - 2\pi i\, \delta(E_{\beta n} - E_{\alpha m}) \langle \Phi_{\beta, n} | U_{\beta\alpha}(E_{\alpha m} + i0) | \Phi_{\alpha, m} \rangle \tag{2.13}$$

shows that the S-matrix is given by the "on-the-energy-shell" matrix elements of the "transition operators" $U_{\beta\alpha}(z)$.

With the help of the resolvent eqs. (2.7) or (2.8), one immediately derives that they fulfill the Faddeev-like equations

$$U_{\beta\alpha} = -(1 - \delta_{\beta\alpha})(H_0 - z) - \sum_{\gamma \neq \beta} T_\gamma G_0 U_{\gamma\alpha} , \tag{2.14}$$

or

$$U_{\beta\alpha} = -(1 - \delta_{\beta\alpha})(H_0 - z) - \sum_{\delta \neq \alpha} U_{\beta\delta} G_0 T_\delta , \tag{2.15}$$

which contain the potentials only implicitly in the transition operators of the two-particle subsystems

$$T_\alpha = V_\alpha - V_\alpha G_\alpha V_\alpha . \tag{2.16}$$

The $U_{\beta\alpha}$ are related to the operators

$$M_{\beta\alpha} = \delta_{\beta\alpha} V_\alpha - V_\beta G V_\alpha \tag{2.17}$$

considered by Faddeev *, according to

$$U_{\beta\alpha} = -(1 - \delta_{\beta\alpha})(H_0 - z) + \sum_{\gamma \neq \beta} \sum_{\delta \neq \alpha} M_{\gamma\delta} . \tag{2.18}$$

With respect to these operators, our $U_{\beta\alpha}$ have the advantage of leading more directly to the desired S-matrix elements, viz. by (2.13). We note that the transition operators $U_{\beta\alpha}^{(\pm)}$ used, e.g., by Lovelace [7, 8] have the same advantage. Being related to the $U_{\beta\alpha}$ by

$$\begin{aligned} U_{\beta\alpha} &= -(1 - \delta_{\beta\alpha})(H_\alpha - z) + U_{\beta\alpha}^{(+)} \\ &= -(1 - \delta_{\beta\alpha})(H_\beta - z) + U_{\beta\alpha}^{(-)} , \end{aligned} \tag{2.19}$$

they coincide with them on the energy shell. Thus they can be inserted in (2.13) instead of $U_{\beta\alpha}$ without a change of the result. On the other hand, they are less symmetric and fulfill integral equations which in contrast to the original Faddeev equations and to eqs. (2.14) or (2.15) contain the potentials V_α explicitly besides the operators T_α. Since only the on-shell amplitudes

* Cf. ref. [2], eq. (3.7).

are needed to determine the S-matrix we may use the most convenient off-shell continuations of them which the $U_{\beta\alpha}$ appear to be. Indeed, performing calculations with the different operators shows at once the advantage of our choice which results in considerable simplifications. Especially it is demonstrated in sect. 3 that the Faddeev-like eqs. (2.14) and (2.15) can be transformed very naturally to multi-channel two-particle Lippmann-Schwinger equations * which in lowest order of perturbation theory reproduce Lovelace's equations. [The original Lovelace equations given in ref. [8], eq. (3.29), can be derived immediately if the separable approximation used there is incorporated in eq. (2.15)].

We furthermore note that the unitarity relation for $U_{\beta\alpha}$ follows from ref. [7], eq. (132), if both $U_{\beta\alpha}^{(+)}$ and $U_{\beta\alpha}^{(-)}$ are replaced by $U_{\beta\alpha}$. This relation can thus be formulated by use of only *one* operator. In more detail we get for the discontinuity across the right-hand cuts

$$U_{\beta\alpha}(E+i\epsilon) - U_{\beta\alpha}(E-i\epsilon) = -2\pi i\, U_{\beta 0}(E+i\epsilon)\Delta_0(E)U_{0\alpha}(E-i\epsilon) - 2\pi i \sum_{\gamma=1}^{3} \sum_{n} U_{\beta\gamma}(E+i\epsilon)\Delta_{\gamma n}(E)U_{\gamma\alpha}(E-i\epsilon)\,, \tag{2.20}$$

where Δ_0 and $\Delta_{\gamma n}$ are the projection operators onto the free three-particle states and the states with two particles bound the third one being free, respectively [7].

In sect. 3, we often use the momentum-space representation. The notation coincides with the one employed in ref. [8]. We also assume that the total momentum is separated off, i.e., the expressions considered depend only on the centre-of-mass momenta $\boldsymbol{p}_\alpha$ of the two-particle subsystems and the relative momenta $\boldsymbol{q}_\alpha$ of the remaining third particles. Let us furthermore recall [cf. ref. [8], eq. (3.11)] that the transition operator T_α acting on the three-particle space is related to the two-particle T-matrix ** $\langle \boldsymbol{p}_\alpha | \hat{T}_\alpha(z) | \boldsymbol{p}'_\alpha \rangle = T_\alpha(\boldsymbol{p}_\alpha, \boldsymbol{p}'_\alpha; z)$ by

$$\langle \boldsymbol{p}_\alpha, \boldsymbol{q}_\alpha | T_\alpha(z) | \boldsymbol{p}'_\alpha, \boldsymbol{q}'_\alpha \rangle = \delta(\boldsymbol{q}_\alpha - \boldsymbol{q}'_\alpha) \langle \boldsymbol{p}_\alpha | \hat{T}_\alpha(z - \boldsymbol{q}_\alpha^2) | \boldsymbol{p}'_\alpha \rangle\,. \tag{2.21}$$

3. MULTI-CHANNEL TWO-PARTICLE LIPPMANN-SCHWINGER EQUATIONS

The equations (2.14) can be written in the form

$$F = I - ITF. \tag{3.1}$$

Here the F, I and T are matrices, the elements of which represent operators in the Hilbert space

* In this respect it is of importance that eqs. (2.14) or (2.15) show already the symmetric structure of these equations.

** In the following hats on operators indicate that these act in the two-particle space.

$$F_{\beta\alpha} = G_0 U_{\beta\alpha} G_0 \,, \tag{3.2}$$

$$I_{\beta\alpha} = -(1 - \delta_{\beta\alpha}) G_0 \,, \tag{3.3}$$

$$T_{\beta\alpha} = -\delta_{\beta\alpha} T_\alpha \,. \tag{3.4}$$

Eq. (3.1) shows the same algebraic structure as the two-particle resolvent equation. Thus we can transfer the procedure used by Weinberg to handle this equation. Especially we may proceed in a manner similar to eqs. (2.11)-(2.17) of ref. [6].

We start by splitting $\hat{T}_\alpha$ into two parts

$$\hat{T}_\alpha = \hat{T}^{\mathrm{S}}_\alpha + \hat{T}'_\alpha \,, \tag{3.5}$$

where $\hat{T}^{\mathrm{S}}_\alpha$ is a sum of separable terms of the form

$$\hat{T}^{\mathrm{S}}_\alpha(z) = -\sum_m |\alpha, m; z\rangle \, t_{\alpha, m}(z) \, \langle\overline{\alpha, m; z}| \,. \tag{3.6}$$

For each bound state or resonance of the subsystem we shall take one separable term. However, it is important to notice that we may add to $\hat{T}^{\mathrm{S}}_\alpha$ further separable terms in order to get a sufficiently small * non-separable remainder $\hat{T}'_\alpha$.

In general the "form factors" $g_{\alpha, m}(\boldsymbol{p}_\alpha; z) = \langle \boldsymbol{p}_\alpha | \alpha, m; z\rangle$ may depend on z. However, if they correspond to bound states $|\psi_{\alpha, m}\rangle$ belonging to the eigenvalues $\hat{E}_{\alpha m} < 0$, they have to fulfill

$$|\alpha, m; z = \hat{E}_{\alpha m}\rangle = \hat{V}_\alpha |\psi_{\alpha, m}\rangle \,. \tag{3.7}$$

Note that $\langle\overline{\alpha, m; z}|$ is not required to be the adjoint of $|\alpha, m; z\rangle$. Eq. (3.5) implies that in the three-particle space we get the same separation

$$T_\alpha = T^{\mathrm{S}}_\alpha + T'_\alpha \,, \tag{3.8}$$

where, according to eq. (2.21),

$$\langle \boldsymbol{p}_\alpha, \boldsymbol{q}_\alpha | \, T^{\mathrm{S}}_\alpha(z) \, | \boldsymbol{p}'_\alpha, \boldsymbol{q}'_\alpha \rangle \tag{3.9}$$

$$= -\delta(\boldsymbol{q}_\alpha - \boldsymbol{q}'_\alpha) \sum_m g_{\alpha, m}(\boldsymbol{p}_\alpha; z - \boldsymbol{q}^2_\alpha) \, t_{\alpha, m}(z - \boldsymbol{q}^2_\alpha) \, \bar{g}^*_{\alpha, m}(\boldsymbol{p}'_\alpha; z - \boldsymbol{q}^2_\alpha) \,.$$

In writing $\{-\sum_m |\alpha, m; z\rangle t_{\alpha, m}(z) \langle\overline{\alpha, m; z}|\}$ in the following it is always to be understood that its momentum-space representation has the form (3.9).

The splitting (3.8) now allows to write (3.1) in the form

$$F = I - I T^{\mathrm{S}} F - I T' F \,, \tag{3.10}$$

where T^{S} and T' are matrices given by (3.4) with T^{S}_α and T'_α instead of T_α, respectively. If we define F' as the solution of

* More precisely, due to the fact that $\hat{T}'_\alpha(z) \hat{G}_0(z)$ is a Hilbert-Schmidt operator for square-integrable potentials and Im $z \neq 0$, the Schmidt norm of $\hat{T}'_\alpha(z) \hat{G}_0(z)$ can be made arbitrarily small.

$$F' = I - IT'F' \,, \tag{3.11}$$

we have, multiplying this equation with $(1 - T^S F)$ from the right,

$$\widetilde{F} = I - IT^S F - IT'\widetilde{F} \,. \tag{3.12}$$

Here the abbreviation

$$\widetilde{F} = F'(1 - T^S F) \tag{3.13}$$

has been used. Comparing (3.12) with (3.10), we conclude

$$\widetilde{F} = F \,, \tag{3.14}$$

which implies the important result

$$F = F' - F' T^S F \,. \tag{3.15}$$

Inserting the definitions (3.2)-(3.4) in (3.15) and remembering (3.6) or (3.9) yields the following integral equation for $U_{\beta\alpha}$:

$$G_0 U_{\beta\alpha} G_0 = G_0 U'_{\beta\alpha} G_0 - \sum_{\gamma,\nu} G_0 U'_{\beta\gamma} G_0 |\gamma,\nu\rangle\, t_{\gamma,\nu} \langle\tilde{\gamma},\tilde{\nu}|\, G_0 U_{\gamma\alpha} G_0 \,, \tag{3.16}$$

where according to (3.11) $U'_{\beta\alpha}$ is given by

$$U'_{\beta\alpha} = -(1 - \delta_{\beta\alpha})(H_0 - z) - \sum_{\gamma\neq\beta} T'_\gamma G_0 U'_{\gamma\alpha} \,. \tag{3.17}$$

Let us now consider the case $\beta \neq 0$, i.e. the case of elastic or re-arrangement collisions *. Defining the "transition amplitudes"

$$X_{\beta n,\alpha m}(z) = \langle \overline{\beta, n; z}\,|\, G_0(z) U_{\beta\alpha}(z) G_0(z)\,|\, \alpha, m; z\rangle \,, \tag{3.18}$$

we find from (3.16) (see fig. 1)

$$X_{\beta n,\alpha m} = Z_{\beta n,\alpha m} - \sum_{\gamma,\nu} Z_{\beta n,\gamma\nu} t_{\gamma\nu} X_{\gamma\nu,\alpha m} \,, \tag{3.19}$$

where the generalized "potentials" $Z_{\beta n,\alpha m}$ are given by (see fig. 2)

$$Z_{\beta n,\alpha m}(z) = \langle \overline{\beta, n; z}\,|\, G_0(z) U'_{\beta\alpha}(z) G_0(z)\,|\, \alpha, m; z\rangle \,. \tag{3.20}$$

If, on the other hand, β designates the channel of three free particles ($\beta = 0$, break-up processes), we introduce the transition amplitudes

$$X_{\boldsymbol{p}_1,\alpha m}(z) = -\langle \boldsymbol{p}_1\,|\, U_{0\alpha}(z) G_0(z)\,|\, \alpha, m; z\rangle \tag{3.21}$$

and the expressions (see fig. 3)

$$Z_{\boldsymbol{p}_1,\alpha m}(z) = -\langle \boldsymbol{p}_1|\, U'_{0\alpha}(z) G_0(z)\,|\, \alpha, m; z\rangle \tag{3.22}$$

to derive again from (3.16) an equation analogous to (3.19) (see fig. 4):

$$X_{\boldsymbol{p}_1,\alpha m} = Z_{\boldsymbol{p}_1,\alpha m} - \sum_{\gamma,\nu} Z_{\boldsymbol{p}_1,\gamma\nu} t_{\gamma\nu} X_{\gamma\nu,\alpha m} \,. \tag{3.23}$$

* Recall the third footnote in sect. 2.

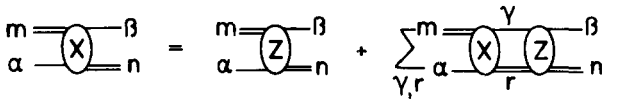

Fig. 1. The integral equation (3.19) for the elastic and re-arrangement scattering amplitudes.

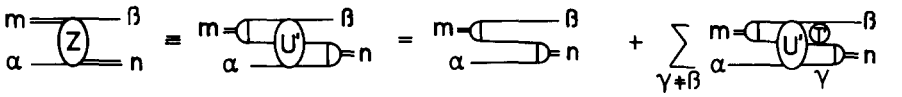

Fig. 2. Graphical form for the integral equation (3.17) for the generalized "potentials" (the semicircles indicate the form factors).

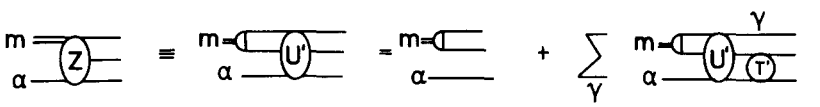

Fig. 3. The expressions $Z_{\boldsymbol{p}_1,\alpha m}$ to be determined from eqs. (3.22) and (3.17) in graphical form.

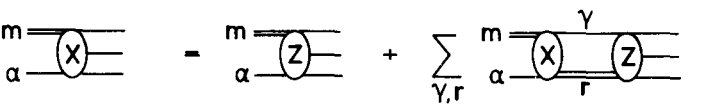

Fig. 4. Graphical representation of eq. (3.23) that yields the break-up amplitude by means of the bound-state scattering amplitudes.

Of course, this is *not* an integral equation. Instead, it yields immediately the break-up amplitudes once the amplitudes for elastic and re-arrangement collisions are calculated from the integral equation (3.19).

It should be stressed that the quantities occurring in eqs. (3.19) and (3.23) are still operators * acting on the plane wave states $|\boldsymbol{q}_\alpha\rangle$. We recognize that (3.19) has the form of a multi-channel two-particle Lippmann-Schwinger equation, the $Z_{\beta n,\alpha m}$ and the $t_{\gamma r}$ playing the role of potentials and of free propagators, respectively. Furthermore we note that for $|\alpha, m; z\rangle$ and $|\beta, n; z\rangle$ belonging to bound states, according to (3.7) we get

$$\delta(E_{\beta n} - E_{\alpha m})\langle \Phi_{\beta,n} | U_{\beta\alpha}(E_{\alpha m} + i0) | \Phi_{\alpha,m}\rangle$$

$$= \delta(E_{\beta n} - E_{\alpha m})\langle \boldsymbol{q}_\beta | X_{\beta n,\alpha m}(E_{\alpha m} + i0) | \boldsymbol{q}_\alpha\rangle, \qquad (3.24)$$

with $E_{\alpha m} = \boldsymbol{q}_\alpha^2 + \hat{E}_{\alpha m}$, $\hat{E}_{\alpha m} < 0$. (A similar result holds, of course, in the case $\beta = 0$.) That is, the $X_{\beta n,\alpha m}$ (or $X_{\boldsymbol{p}_1,\alpha m}$) yield immediately the transition amplitudes necessary for the calculation of the S-matrix from eq. (2.13). In other words, the three-particle collision problem is reduced to the solution of the two-particle Lippmann-Schwinger equations (3.19) which,

* Recall especially eq. (3.9).

after angular momentum decomposition, represent a system of one-dimensional integral equations. The following difference with the two-particle case should be noted. Here the application of the Schmidt method yields purely algebraic equations for the transition amplitudes. We are left with one-dimensional integral equations in the three-particle case because the splitting of the kernel into separable and non-separable parts is only performed in the subsystem transition operators; i.e., we have not taken into account that the square of the Faddeev kernel is a Hilbert-Schmidt operator which could be approximated by terms which are completely separable in all the momentum variables *.

In order to find the generalized potentials needed in eqs. (3.19) and (3.23), we have to solve eq. (3.17) which shows the same complicated structure as the original Faddeev-like eq. (2.14). However, because eq. (3.17) contains only the non-separable part T'_α of T_α we may approximate $U'_{\beta\alpha}$ by iterating this equation. We recall [cf. the discussion following eq. (3.6)] that T'_α can be made small enough to assure rapid convergence of this iteration.

In lowest order we get **

$$U'^{(0)}_{\beta\alpha}(z) = -(1-\delta_{\beta\alpha})(H_0 - z) \ , \tag{3.25}$$

i.e.

$$Z^{(0)}_{\beta n,\,\alpha m}(z) = -(1-\delta_{\beta\alpha})\langle \overline{\beta, n; z}\,|\, G_0(z)\,|\alpha, m; z\rangle \ , \tag{3.26}$$

$$Z^{(0)}_{\boldsymbol{p}_1,\,\alpha m}(z) = \langle \boldsymbol{p}_1\,|\alpha, m; z\rangle \ , \tag{3.27}$$

just reproducing the Lovelace equations [ref. [8], eq. (3.29)].

The next step in the interation yields for $U'_{\beta\alpha}$ an additional term

$$\sum_{\gamma\neq\alpha,\beta} T'_\gamma(z) \ .$$

A second iteration results in

$$Z^{(2)}_{\beta n,\,\alpha m} = -(1-\delta_{\beta\alpha})\langle \overline{\beta, n}\,|G_0\,|\alpha, m\rangle + \sum_{\gamma\neq\alpha,\beta} \langle \overline{\beta, n}\,|G_0 T'_\gamma G_0\,|\alpha, m\rangle$$

$$- \sum_{\gamma\neq\beta} \sum_{\rho\neq\alpha,\gamma} \langle \overline{\beta n}\,|G_0 T'_\gamma G_0 T'_\rho G_0\,|\alpha, m\rangle \ , \tag{3.28}$$

$$Z^{(2)}_{\boldsymbol{p}_1,\,\alpha m} = \langle \boldsymbol{p}_1\,|\alpha, m\rangle - \sum_{\gamma\neq\alpha} \langle \boldsymbol{p}_1\,|T'_\gamma G_0\,|\alpha, m\rangle$$

$$+ \sum_{\gamma} \sum_{\rho\neq\alpha,\gamma} \langle \boldsymbol{p}_1\,|T'_\gamma G_0 T'_\rho G_0\,|\alpha, m\rangle \ . \tag{3.29}$$

* To proceed in the latter way would represent an alternative version of application of the Schmidt method [6]. Since it is rather difficult to find good completely separable approximations, we did not follow this procedure.

** In this approximation the inhomogeneous term of (3.23) vanishes on the energy shell, of course.

(The approximations (3.28) and (3.29) are depicted in figs. 5 and 6.)

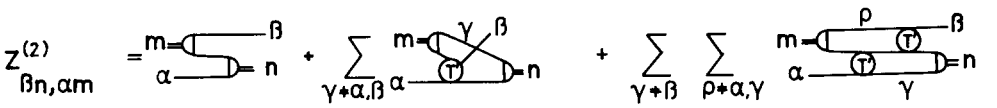

Fig. 5. The generalized potentials approximated up to second order [cf. eq. (3.28)].

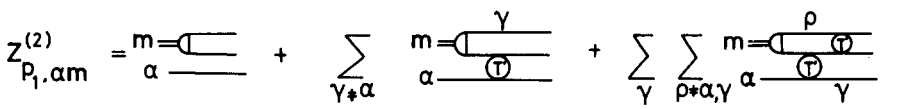

Fig. 6. Eq. (3.29) in graphical form.

4. THREE-PARTICLE BOUND STATES

An integral equation for the three-particle, bound states $|\Psi_\nu\rangle$, belonging to the eigenvalues E_ν, can easily be derived from (3.16). Due to the fact that $U_{\beta\alpha}$ has a pole for $z = E_\nu < 0$, it follows that

$$G_0(E_\nu)\overline{V}_\alpha|\Psi_\nu\rangle = -G_0(E_\nu)\sum_{\gamma,r} U'_{\alpha\gamma}(E_\nu)G_0(E_\nu)\,|\gamma,r;E_\nu\rangle \times t_{\gamma,r}(E_\nu)\langle\overline{\gamma,r;E_\nu}|\,G_0(E_\nu)\overline{V}_\gamma\,|\Psi_\nu\rangle\,. \tag{4.1}$$

Let us define

$$|Q_\nu^{\alpha,m}\rangle = t_{\alpha,m}(E_\nu)\langle\overline{\alpha,m;E_\nu}\,|\,G_0(E_\nu)\overline{V}_\alpha|\Psi_\nu\rangle\,, \tag{4.2}$$

which is still a vector in the Hilbert space spanned by the plane wave states $|\boldsymbol{q}_\alpha\rangle$. Inserting this definition into (4.1), we find (take $\alpha = 0$ and recall $\overline{V}_0 = V$)

$$|\Psi_\nu\rangle = \sum_{\gamma,r} G_0(E_\nu)U'_{0\gamma}(E_\nu)G_0(E_\nu)\,|\gamma,r;E_\nu\rangle|Q_\nu^{\gamma,r}\rangle\,. \tag{4.3}$$

Furthermore, from (4.1) and (4.2), we derive

$$|Q_\nu^{\alpha,m}\rangle = -t_{\alpha,m}(E_\nu)\sum_{\gamma,r} Z_{\alpha m,\gamma r}(E_\nu)\,|Q_\nu^{\gamma,r}\rangle\,. \tag{4.4}$$

This integral equation fixes the eigenvalues E_ν and the vectors $|Q_\nu^{\alpha,m}\rangle$ which yield, by means of (4.3), the states $|\Psi_\nu\rangle$. In other words, the three-particle bound-state problem is reduced in the same manner as the scattering problem to a one-dimensional integral equation. Inserting the lowest-order approximation for $Z_{\alpha m,\beta n}$, given by (3.26), we reproduce a result given by Phillips for separable potentials [cf. ref. [10], eq. (4.6]. But of course, as in the scattering case, our formalism allows for non-separable

corrections to the calculation of the bound-state energy.

The considerations of this section and of the previous one are based on the splitting of the two-particle transition operators into their separable and non-separable parts. This procedure is, of course, not unique. Several forms have been proposed, e.g. by Lovelace, for the choice of the separable parts $\hat{T}^{S}_{\alpha}(z)$ [cf. ref. [8], eqs. (2.54), (2.63) and (2.64)]. A more systematical treatment has been put forward by Weinberg [3-5] using the Schmidt method. As we have already pointed out, this procedure is formally identical with ours if we replace F, I and T in (3.1) by the two-particle operators $\hat{G}$, $\hat{G}_0$ and $\hat{V}$, respectively. By this method we get in the general case instead of (3.6) the more complicated form

$$\hat{T}^{S}_{\alpha}(z) = - \sum_{m,n} |\alpha, m; z\rangle t_{\alpha, mn}(z) \langle\overline{\alpha, n; z}| , \tag{4.5}$$

which, nevertheless, by a slight modification can be incorporated in our formalism, developed in sect. 3. Note that in the "ideal choice" [4] eq. (4.5) reduces to eq. (3.6).

Other methods for splitting $\hat{V}$ into a separable and a non-separable part have been proposed by Sugar and Blankenbecler [11], by Noyes [12], by Kowalski [13] and by Mongan [14]. Finally we note that three-body forces can easily be incorporated in our equations in a manner described in refs. [10, 15].

5. SPECIAL CASES

We would like to discuss now two special cases of our eqs. (3.19) and (3.23) for which the equations for the "potentials" become very simple

(i) Let one two-body amplitude be formally split into a separable and a non-separable part (for definiteness we call this channel $\alpha = 1$). The remaining two amplitudes are supposed to be non-separable $T_\beta = T'_\beta$, $\beta \neq \alpha$, and "small" in the sense that it might be reasonable to keep only first order terms of them *. Thus, the relevant potentials are given by eqs. (3.28) and (3.29). Therefore the scattering amplitude becomes very simple (fig. 7a)

$$X_{1n,1m}(z) \approx \sum_{\gamma=2}^{3} \langle\overline{1, n; z}| G_0(z) T'_\gamma(z) G_0(z) |1, m; z\rangle . \tag{5.1}$$

Since, as already mentioned, $\langle \boldsymbol{q}_1 | Z^{(0)}_{\boldsymbol{p}_1, \alpha m} | \boldsymbol{q}_\alpha \rangle$ vanishes on the energy shell, we find for the on-shell break-up amplitude (see fig. 7b).

* An interesting example of this situation is provided by the elastic and inelastic electron-deuteron scattering.

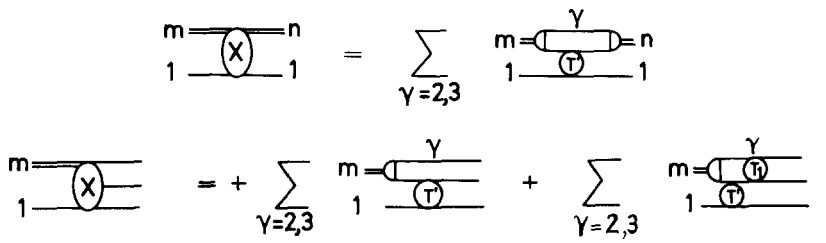

Fig. 7. Graphical representation of a) the bound-state scattering amplitudes (5.1), b) the break-up amplitude (5.2).

$$
\begin{aligned}
X_{\boldsymbol{p}_1,1m}(z) \approx & - \sum_{\gamma\neq 1} \langle \boldsymbol{p}_1 | T'_\gamma(z) G_0(z) | 1, m; z\rangle \\
& \sum_{\gamma\neq 1} \sum_{r} g_{1,r}(\boldsymbol{p}_1; z)\, t_{1,r}(z) \langle \overline{1, r; z} | G_0(z) T'_\gamma(z) G_0(z) | 1, m; z\rangle \\
& + \sum_{\gamma\neq 1} \langle \boldsymbol{p}_1 | T'_1(z) G_0(z) T'_\gamma(z) G_0(z) | 1, m; z\rangle \\
= & - \sum_{\gamma\neq 1} \langle \boldsymbol{p}_1 | T'_\gamma(z) G_0(z) | 1, m; z\rangle \\
& + \sum_{\gamma\neq 1} \langle \boldsymbol{p}_1 | T_1(z) G_0(z) T'_\gamma(z) G_0(z) | 1, m; z\rangle \ . \qquad (5.2)
\end{aligned}
$$

Here T_1 is the full amplitude of the $\alpha = 1$ subsystem. Note that this result is correct up to first order in the small amplitudes T'_2 and T'_3. It leads to a type of spectator model with final state interaction. [This result also follows, of course, directly from eq. (2.14).]

(ii) Let us now assume that all two-body potentials are separable, one of them, viz. V_3, contains an additional non-separable part V'_3. To all orders in the corresponding amplitude T'_3, the generalized potentials are given by [cf. eqs. (3.28) and (3.29)]

$$
\begin{aligned}
Z_{\beta n,\alpha m}(z) = & - (1 - \delta_{\beta\alpha}) \langle \overline{\beta, n; z} | G_0(z) | \alpha, m; z\rangle \\
& + (1 - \delta_{\beta 3})(1 - \delta_{\alpha 3}) \langle \overline{\beta, n; z} | G_0(z) T'_3(z) G_0(z) | \alpha, m; z\rangle \ , \qquad (5.3)
\end{aligned}
$$

$$
Z_{\boldsymbol{p}_1,\alpha m}(z) = \langle \boldsymbol{p}_1 | \alpha, m; z\rangle - (1 - \delta_{\alpha 3}) \langle \boldsymbol{p}_1 | T'_3(z) G_0(z) | \alpha, m; z\rangle \ . \qquad (5.4)
$$

They are graphically shown in figs. 8a and b, respectively. Inserting these expressions in (3.19), we get integral equations for the elastic and rearrangement amplitudes which are depicted in fig. 9a. The corresponding (on-shell) break-up amplitude given by (3.23) is shown in fig. 9b. These

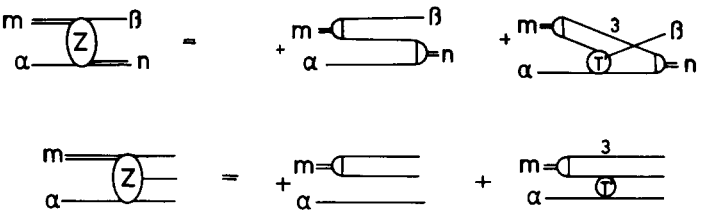

Fig. 8. The exact generalized "potentials" when $T'_1 = T'_2 = 0$, a) for bound-state scattering, b) for composite particle disintegration, (cf. eqs. (5.3) and (5.4), respectively)

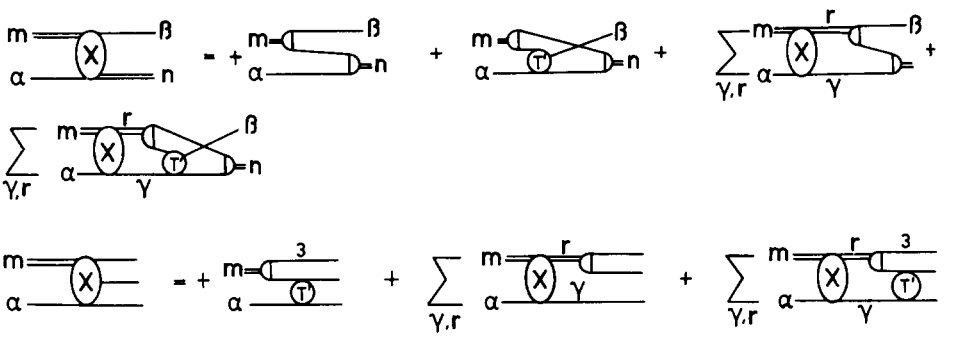

Fig. 9. Diagrams for a) the bound-state scattering amplitude and b) the break-up amplitude discussed in subsect. 5(ii).

equations permit e.g. to incorporate Coulomb corrections to proton-deuteron scattering which had to be neglected [10] previously in calculations with the Lovelace equations.

6. CONCLUSIONS

We have proposed a systematical and exact treatment of three-particle bound-state scattering including break-up reactions. One-dimensional Lippmann-Schwinger-type integral equations have been written down for the relevant amplitudes. The generalized potentials occurring in them are to be found by solving Faddeev-type equations which, however, can be approximated by a few terms of a Born series expansion; e.g. whereas the original Faddeev equations, though solvable in principle, reveal nearly unsurmountable calculational difficulties, our approach admits the use of the methods of perturbation theory. Depicted in form of diagrams, the physical meaning of our successive approximations becomes transparent. Having determined the generalized potentials, our one-dimensional integral equations can be solved by standard methods. In particular, the Schmidt method could be applied once more to reduce them to purely algebraic equations.

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