Derivation of a formula for the $A(d, p)B$ transition amplitude from the Faddeev equations of three-body scattering theory

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I rederive the formula for the $B(p, d)A$ and $A(d, p)B$ transition amplitudes used by Timofeyuk and Johnson [Phys. Rev. C **59**, 1545 (1999)] in their study of reactions involving halo nuclei. The method of derivation is to use the coupled equations for the three-body problem in the version of the Faddeev equations reported by Alt, Grassberger, and Sandhas [Nucl. Phys. **B2**, 167 (1967)].

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I. INTRODUCTION

The aim of this work is to make a connection in the context of $A(d, p)B$ and $B(p, d)A$ reactions between the three-body formalism of Alt, Grassberger, and Sandhas [1] and the formalism of distorted-wave potentials and three-body wave functions customarily used in nuclear reaction theory.

In the theory of Alt *et al.* [1] transfer amplitudes appear as the matrix elements between plane-wave channel states of a set of operators $U_{\alpha,\beta}$, where α and β label all possible channels in the three-body system. These operators satisfy coupled integral equations that are particularly convenient when an exact solution of the three-body Schrödinger equation is required. A major recent advance is the extension of this approach so that charged particles can be handled in a practical way [2], and this has increased the interest in applications to nuclear reaction theory [3].

Three-body models of nuclear reactions involving deuterons and massive nuclei have proved invaluable in increasing our understanding of the key role of deuteron breakup effects, but we are usually interested in extracting nuclear structure information from data in a credible fashion and therefore a key issue is the link between the three-body models and the underlying many-body problem. Three-body-model Hamiltonians will always be an approximate image of the many-body system. From the point of view of the nuclear structure practitioner, therefore, we must learn how to use the exact solutions of three-body models to provide guidance on how deuteron breakup effects can be included in a way that can be generalized to the many-body case. This means, for example, that emphasis is placed on determining an adequate representation of the three-body scattering wave function in restricted regions of the six-dimensional configuration space rather than calculating an accurate version everywhere; for the latter purpose the coupled equations of Alt *et al.* [1] are well adapted.

The 1999 paper by Timofeyuk and Johnson [4] was motivated along these lines. Thus they used a particular formula for the (d, p) and (p, d) transition amplitude that enabled breakup effects to be taken into account approximately. At the same time, because their formula did not contain the "remnant term" that appears in standard many-body theories of deuteron stripping and pickup (see, e.g., Ref. [5], p. 151), there was a transparent link to many-body concepts such as overlap functions, spectroscopic factors, and asymptotic normalization factors [6,7]. In this paper we show how this particular formula can be derived from the coupled equations of Alt *et al.* [1].

The three-body problem of interest refers to p, n, and a target nucleus A. We do not use the traditional "odd man out" notation of three-body theory because we are only interested in the operator for the transition between a single pair of channels: channel p, in which the proton is far from a bound state, B, of n and A; and channel d, in which A is far from a bound state, d, of n and p. The relevant transition operator is defined by [1]

$$U_{pd} = G_0^{-1} + V_{pA} + (V_{pA} + V_{pn})G(V_{pA} + V_{nA}), \qquad (1)$$

where V_{pn} is the proton-neutron interaction, V_{nA} the neutron-A interaction (assumed to be real for the purposes of this article), and V_{pA} the proton-A interaction. All Coulomb interactions are assumed to be screened at large distances. $G_0(z) = (z - K)^{-1}$ and $G(z) = (z - K - V)^{-1}$ are the Green functions associated with the free and the full three-body Hamiltonians, respectively. K is the total kinetic energy operator and V is the total potential energy. The dependence on the complex energy parameter z has been suppressed in Eq. (1) for simplicity.

Our aim is to prove that the operator U_{pd} satisfies

$$U_{pd} = \bar{U}_{pd} G_0 V_{np} G_p U_{pd}, \qquad (2)$$

where $\bar{U}_{pd}$ is what U_{pd} reduces to when V_{np} vanishes and $G_p(z) = (z - K - V_{nA})^{-1}$ is the Green function appropriate to the asymptotic Hamiltonian in the outgoing proton channel.

The identity is valid when both sides operate on a plane-wave deuteron state,

$$\phi_{\vec{k}_d,d} = \exp(\imath \vec{k}_d \cdot \vec{R})\, \phi_d(\vec{r}), \qquad (3)$$

and for an energy parameter $E + \imath\varepsilon$, $\varepsilon \to 0^+$, with $E = (\hbar^2 k_d^2 2m_{dA}) - \epsilon_d$, that is, on-shell.

The operator $\bar{U}_{pd}$ is defined by

$$\bar{U}_{pd} = G_0^{-1} + V_{pA} + V_{pA}\bar{G}(V_{pA} + V_{nA}), \qquad (4)$$

where $\bar{G}$ is the Green function corresponding to a Hamiltonian with $V_{np} = 0$ everywhere.

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An alternative expression for $\bar{U}_{pd}$ is

$$\bar{U}_{pd} G_0 = 1 + V_{pA} \bar{G}. \tag{5}$$

This identity and Eq. (2) are proved in Sec. III, but first we discuss the significance of these results.

II. DISCUSSION

There are several features of the result, Eq. (2), that are important for practical applications to nuclear reactions. At first glance formula (2) does not appear to be very useful because the operator U_{pd} appears on both sides of the equation. We see in the following that an important advantage is that the right-hand-side expression emphasizes a particular projection of U_{pd} where the neutron and proton are within the range of the short-range potential V_{np}.

(i) The operator $\bar{U}_{pd}$ has a simple significance when it acts on a proton-channel plane wave,

$$\phi_{\vec{k}_p,B} = \exp(\imath \vec{k}_p \cdot \vec{r}_{pB}) \phi_B(\vec{r}_{nA}), \tag{6}$$

and when $m_n/m_B \to 0$. In this limit the kinetic energy operator separates in the coordinates $\vec{r}_{pA}$ and $\vec{r}_{nB}$ and we find

$$\begin{aligned}\langle \phi_{\vec{k}_p,B} | \bar{U}_{pd} G_0 &= \langle \phi_{\vec{k}_p,B} | \left[1 + \left(V_{pA} \frac{1}{E^+ - K_{nA} - V_{nA} - K_{pA} - V_{pA}} \right) \right] \\ &= \langle \phi_{\vec{k}_p,B} | \left[1 + \left(V_{pA} \frac{1}{E_p^+ - K_{pA} - V_{pA}} \right) \right] \\ &= \langle \chi^{(-)}_{\vec{k}_p} \phi_B |, \end{aligned} \tag{7}$$

where $\chi^{(-)}_{\vec{k}_p}(\vec{r}_{pA})$ is a distorted-wave solution of the two-body Schrödinger equation,

$$(E - K_{pA} - V^*_{pA}) \chi^{(-)}_{\vec{k}_p}(\vec{r}_{pA}) = 0, \tag{8}$$

and ϕ_B satisfies

$$(-\epsilon_B - K_{nA} - V_{nA}) \phi_B(\vec{r}_{nA}) = 0. \tag{9}$$

The proton energy E_p and the neutron binding energy ϵ_B are related to the total energy E by $E = E_p - \epsilon_B$.

For finite values of m_n/m_B the operator $\bar{U}_{pd}$ is the solution of a special three-body problem with one of the two-body interactions absent. For practical applications to many nuclear reactions of interest the ratio m_n/m_B is small and perturbation methods with m_n/m_B as the expansion parameter may be convenient. It has been argued elsewhere [6,7] that the application of continuum-discretized coupled-channels (CDCC) methods to the evaluation of $\bar{U}_{pd}$ may not be appropriate in the context of the evaluation of Eq. (1), although CDCC methods may be quite appropriate for the evaluation of $V_{np} G_p U_{pd}$.

(ii) Matrix elements on the right-hand side of Eq. (2) between on-shell channel states express the (d, p) transfer amplitude in exactly the form used as the starting point of the analysis of (d, p) and (p, d) reactions by Timofeyuk and Johnson [4]. To see this we use the on-shell identity (with the limit $\varepsilon \to 0^+$ understood),

$$G_p U_{pd} |\phi_{\vec{k}_d,d}\rangle = |\Psi^{(+)}_{\vec{k}_d,d}\rangle, \tag{10}$$

where $\Psi^{(+)}_{\vec{k}_d,d}$ is the exact solution of the three-body Schrödinger equation corresponding to an incident plane-wave deuteron and outgoing scattered waves in all channels. This result follows from one of the relations between G and U_{pd} given in Ref. [1]:

$$G = G_p U_{pd} G_d, \tag{11}$$

where G_d is the Green function appropriate to the asymptotic Hamiltonian in the deuteron channel. The result, Eq. (10), then follows from the on-shell result,

$$\imath\varepsilon G_d |\phi_{\vec{k}_d,d}\rangle = |\phi_{\vec{k}_d,d}\rangle, \tag{12}$$

and the standard (see, e.g., Ref. [8], p. 176) expression for $\Psi^{(+)}_{\vec{k}_d,d}$ in terms of G,

$$|\Psi^{(+)}_{\vec{k}_d,d}\rangle = \imath\varepsilon G |\phi_{\vec{k}_d,d}\rangle, \tag{13}$$

in the limit $\varepsilon \to 0^+$.

A consequence of formula (2) is, therefore,

$$\begin{aligned}\langle \phi_{\vec{k}_p,B} | U_{pd} | \phi_{\vec{k}_d,d} \rangle &= \langle \phi_{\vec{k}_p,B} | \bar{U}_{pd} G_0 V_{np} G_p U_{pd} | \phi_{\vec{k}_d,d} \rangle \\ &= \langle \bar{\Psi}^{(-)}_{\vec{k}_p,B} | V_{np} | \Psi^{(+)}_{\vec{k}_d,d} \rangle, \end{aligned} \tag{14}$$

where $\langle \bar{\Psi}^{(-)}_{\vec{k}_p,B}|$ is defined by

$$\langle \bar{\Psi}^{(-)}_{\vec{k}_p,B} | = \langle \phi_{\vec{k}_p,B} | (1 + V_{pA} \bar{G}) \tag{15}$$

and satisfies the differential equation,

$$(E - K - V_{nA} - V^*_{pA}) \bar{\Psi}^{(-)}_{\vec{k}_p,B} = 0. \tag{16}$$

As already explained, when $m_n/m_B \to 0$ the function $\bar{\Psi}^{(-)}_{\vec{k}_p,B}$ has the factored form given in Eqs. (7)–(9) and does not require the solution of a three-body problem.

Formula (14) for the (d, p) transition amplitude, as well as the analogous formula for (p, d), was used by Timofeyuk and Johnson [4]. This has the advantage that it requires accurate knowledge of the scattering wave function inside the range of V_{np} only where CDCC methods may be appropriate [4,7]. The result (14) was also derived by Goldberger and Watson [8], pp. 833–841, using manipulations of the Lippmann-Schwinger equation for $|\Psi^{(+)}_{\vec{k}_d,d}\rangle$, but only in the $m_n/m_B \to 0$ limit.

III. PROOFS

A. Proof of Eq. (5)

From the definition (4) we have

$$\begin{aligned}\bar{U}_{pd} G_0 &= 1 + V_{pA} G_0 + V_{pA} \bar{G} (V_{pA} + V_{nA}) G_0 \\ &= 1 + V_{pA} G_0 + V_{pA} (\bar{G} - G_0) \\ &= 1 + V_{pA} \bar{G}, \end{aligned} \tag{17}$$

which is the required result.

In going from the first to the second line in Eq. (17) we have used the operator identity

$$\begin{aligned}\bar{G} - G_0 &= \bar{G}\left(G_0^{-1} - \bar{G}^{-1}\right)G_0 \\ &= \bar{G}[(z-K) - (z - K - V_{nA} - V_{pA})]G_0 \\ &= \bar{G}(V_{nA} + V_{pA})G_0. \end{aligned} \tag{18}$$

B. Proof of Eq. (2)

Starting from Eq. (1) for U_{pd} we first extract the part that is independent of V_{np} so that

$$\begin{aligned} U_{pd} &= G_0^{-1} + V_{pA} + V_{pA}G(V_{pA} + V_{nA}) \\ &\quad + V_{pn}G(V_{pA} + V_{nA}) \\ &= G_0^{-1} + V_{pA} + V_{pA}(\bar{G} + \bar{G}V_{np}G)(V_{pA} + V_{nA}) \\ &\quad + V_{pn}G(V_{pA} + V_{nA}) \\ &= \bar{U}_{pd} + V_{pA}\bar{G}V_{np}G(V_{pA} + V_{nA}) \\ &\quad + V_{pn}G(V_{pA} + V_{nA}) \\ &= \bar{U}_{pd} + (V_{pA}\bar{G} + 1)V_{np}G(V_{pA} + V_{nA}), \end{aligned} \tag{19}$$

where Eq. (4) has been used for $\bar{U}_{pd}$. In going from the second to the third line we have used a formula analogous to Eq. (18) for the difference $G - \bar{G}$.

Using the identity, Eq. (5), we can replace the last line in Eq. (19) with

$$U_{pd} = \bar{U}_{pd}[1 + G_0 V_{np} G(V_{pA} + V_{nA})]. \tag{20}$$

From the relation between $G = G_p U_{pd} G_d$ given in Ref. [1], Eq. (2.11), we deduce

$$\begin{aligned} G_p U_{pd} &= G G_d^{-1} = G\left(G_d^{-1} - G^{-1}\right) + 1 \\ &= G(V_{nA} + V_{pA}) + 1, \end{aligned} \tag{21}$$

so that Eq. (20) can be written

$$\begin{aligned} U_{pd} &= \bar{U}_{pd}[1 + G_0 V_{np}(G_p U_{pd} - 1)] \\ &= \bar{U}_{pd} G_0 V_{np} G_p U_{pd} + \bar{U}_{pd}(1 - G_0 V_{np}). \end{aligned} \tag{22}$$

This equation is an operator identity that is valid for any value of the complex energy parameter. It can be regarded as an integral equation for U_{pd} in which the dependence on V_{np} is made explicit.

Acting on the deuteron channel state, Eq. (3), we have

$$\begin{aligned} (1 - G_0 V_{np})|\phi_{\vec{k}_d,\, d}\rangle &= G_0\left(G_0^{-1} - V_{np}\right)\left|\phi_{\vec{k}_d,d}\right\rangle \\ &= G_0\left(G_0^{-1} - V_{np}\right)\left|\phi_{\vec{k}_d,d}\right\rangle \\ &= G_0(E + \imath\varepsilon - K - V_{np})\left|\phi_{\vec{k}_d,d}\right\rangle \\ &= \imath\varepsilon G_0\left|\phi_{\vec{k}_d,d}\right\rangle, \end{aligned} \tag{23}$$

which vanishes when $\varepsilon \to 0^+$. Hence in the same limit the second term on the right-hand side of Eq. (23) does not contribute and the result, Eq. (2), is obtained.

This result is easily generalized to the application of the $U_{\beta\alpha}$ operator for any channels $\alpha \neq \beta$. We find

$$U_{\beta\alpha}|\phi_\alpha\rangle = \bar{U}_{\beta\alpha} G_0 V_\alpha G_\beta U_{\beta\alpha}|\phi_\alpha\rangle, \tag{24}$$

when acting on an on-shell channel state ϕ_α in the limit $\varepsilon \to 0^+$. $U_{\beta\alpha}$ reduces to $\bar{U}_{\beta\alpha}$ when V_α vanishes.

For the case of the deuteron breakup transition amplitude we obtain a formula analogous to Eq. (14). In the limit $m_n/m_B \to 0$ the final-state wave function $\bar{\Psi}^{(-)}_{\vec{k}_p,\vec{k}_n}$ becomes a product of the proton distorted-wave $\chi^{(-)}_{\vec{k}_p}$ with a neutron distorted-wave $\chi^{(-)}_{\vec{k}_n}$ corresponding to the potential V_{nA} instead of the bound state ϕ_B.

IV. CONCLUSIONS

We have provided a link between the formalism used in exact calculations of amplitudes for transfer within three-body models and a matrix element that has been found useful for calculating $A(d,p)B$ and $B(p,d)A$ cross sections on massive nuclei.

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[1] E. O. Alt, P. Grassberger, and W. Sandhas, Nucl. Phys. **B2**, 167 (1967).
[2] A. Deltuva, A. C. Fonseca, and P. U. Sauer, Phys. Rev. C **71**, 054005 (2005); Phys. Rev. Lett. **95**, 092301 (2005); Phys. Rev. C **72**, 054004 (2005); **73**, 057001 (2006).
[3] A. Deltuva, A. M. Moro, E. Cravo, F. M. Nunes, and A. C. Fonseca, Phys. Rev. C **76**, 064602 (2007).
[4] N. K. Timofeyuk and R. C. Johnson, Phys. Rev. C **59**, 1545 (1999).
[5] I. Thompson and F. Nunes, *Nuclear Reactions for Astrophysics* (Cambridge University Press, Cambridge, UK, 2009).
[6] R. C. Johnson, AIP Conf. Proc. **791**, 128 (2005).
[7] A. M. Moro, F. N. Nunes, and R. C. Johnson (submitted for publication, 2009).
[8] M. L. Goldberger and K. M. Watson, *Collision Theory* (Wiley, New York 1964) (reprinted by Dover, New York, 2004).