$$\nabla \cdot \left(\nabla S^{(0)} e^{2S^{(1)}} \right) = 0$$

or

$$\nabla \cdot \left(\bar{p}(r) |\psi|^2 \right) = 0 \tag{48}$$

which, on interpreting

$$\frac{1}{m} \vec{p}(r)$$

as the velocity field and

$$|\psi|^2$$

as the density of particles, is seen to be just the hydrodynamical equation of continuity. Now suppose a beam of particles is projected toward a scattering center. The density of particles, according to the equation, is at all points the density present if the particles simply move along their classical trajectories. Hence cross sections, for example, computed by the W. K. B. method will be precisely the same as those computed classically. (An exception arises only if the potential is sufficiently complicated that more than one classical trajectory can lead to the point of observation. In that case, Eq. (43) must be replaced by a sum of similarly constructed terms, one for each path, and non-classical interference effects arise.)

A number of authors have applied the name of the W. K. B. approximation to a rather different procedure based on the partial wave expansion. They integrate the radial equations to find the individual phase shifts by a one-dimensional W. K. B. method. The region of applicability of this procedure is altogether different from that of the one described above, although, of course, the two overlap at the classical extreme. It is questionable whether such an alternative procedure should be referred to simply as the W. K. B. approximation since its use of exact angular eigenfunctions leads to such different mathematical properties. This approximation is, in fact, related to one which we shall describe presently.

<u>The High-Energy Approximation in One Dimension</u>

We shall now begin the development of an approximation which is better suited to many of the purposes of high-energy studies than any of the methods mentioned earlier. While the method to be discussed is not without limitations of its own, these, as we shall see, allow one to estimate correctly the intensity of a predominant part of the scattering.

To begin the development at the simplest possible point, and one that will later prove quite useful, we shall consider a one-dimensional scattering problem. Of course it is necessary to bear in mind a very special property of scattering problems in one dimension. The scattering process can take place in only two directions, either preserving the sense of motion of the particle or sending it directly backward. There are no compromises. While this makes the problem a trifle unrealistic, it has the advantage of making it mathematically more transparent.

The Schroedinger equation in one dimension is

$$\left(\frac{d^2}{dx^2} + k^2\right)\psi(x) - \frac{2m}{\hbar^2}V(x)\psi(x)\,. \qquad (49)$$

Now we shall assume that the energy of the incident particle greatly exceeds the magnitude of the potential $V(x)$, and is also large enough that the particle wavelength is much smaller than the potential width a

$$\frac{V}{E} \ll 1 \quad , \quad ka \gg 1\,. \qquad (50)$$

(In order of magnitude relations such as this, the symbol V is to be interpreted as a measure of the <u>absolute magnitude</u> of the potential.) Under these conditions we are justified in assuming that back-scattering will be very weak, that the wave function of the particle may to a good approximation be written in the form

$$\psi(x) = e^{ikx}\varphi(x)\,, \qquad (51)$$

where $\varphi(x)$ is a function which varies slowly over a particle wavelength. Substituting into the Schroedinger equation, we secure

$$\left(2ik\frac{d}{dx} + \frac{d^2}{dx^2}\right)\varphi(x) = \frac{2m}{\hbar^2}V(x)\varphi(x)\,. \qquad (52)$$

Now our approximation consists in dropping the

$$\frac{d^2}{dx^2}$$

term since we assume φ varies slowly in a wavelength. In that case the equation reduces to

$$\frac{d\varphi}{dx} = -\frac{i}{\hbar v}V(x)\varphi(x)\,. \qquad (53)$$

Now if Eq. (51) is to reduce to the incident plane wave at $x = -\infty$ (i. e., back-scattering is neglected) we require as a boundary condition $\varphi(-\infty) = 1$. Thus we secure

$$\varphi(x) = e^{-\frac{i}{\hbar v}\int_{-\infty}^{x} V(x')\,dx'} \tag{54}$$

and

$$\psi(x) = e^{ikx - \frac{i}{\hbar v}\int_{-\infty}^{x} V(x)\,dx} \quad . \tag{55}$$

Note that if the exponential, $\varphi(x)$, were expanded in a power series, the successive terms would represent those of the Born approximation series. In this way, we may verify directly that the expansion parameter of the series is

$$Va/\hbar v .$$

The approximation we have just described may be derived in another way which is also fairly instructive. Here we begin with the one-dimensional version of the integral equation for scattering

$$\psi(x) = e^{ikx} + \int G(x-x')V(x')\psi(x')\,dx'. \tag{56}$$

The one-dimensional Green's function we require may be expressed as

$$G(x-x') = -\frac{m}{\pi\hbar^2}\int_{-\infty}^{\infty}\frac{e^{i\lambda(x-x')}}{\lambda^2-k^2-i\epsilon}\,d\lambda \ , \tag{57}$$

where the outgoing wave boundary condition for G requires that we take the limit of this expression as $\epsilon \to 0$ through positive values. The result is simply

$$G(x-x') = -\frac{i}{\hbar v}e^{ik|x-x'|} \quad . \tag{58}$$

We again express $\psi(x)$ in the form

$$\psi(x) = e^{ikx}\varphi(x) \quad ,$$

so that the integral equation for φ becomes

$$\varphi(x) = 1 - \frac{i}{\hbar v}\int_{-\infty}^{\infty} e^{ik|x-x'| - ik(x-x')} V(x')\varphi(x')dx' \tag{59}$$

$$= 1 - \frac{i}{\hbar v}\int_{-\infty}^{x} V(x')\varphi(x')dx' - \frac{i}{\hbar v}\int_{x}^{\infty} e^{2ik(x-x')} V(x')\varphi(x')dx'. \tag{60}$$

When the two regions of integration $x' < x$ and $x' > x$ are separated, their integrands are seen to vary in altogether different ways. Now if the functions $V(x)$ and φ both vary slowly in a particle wavelength, the rapidly oscillating exponential in the second integrand may be expected to reduce its contribution considerably in magnitude. As a first approximation, therefore, we shall neglect the integral over the region $x' > x$. It is clear from the form of this integral that we are thereby neglecting back-scattering. The integral equation which remains is simply

$$\varphi(x) = 1 - \frac{i}{\hbar v}\int_{-\infty}^{x} V(x')\varphi(x')dx', \tag{61}$$

which may be solved trivially by differentiating, so that we are again led to the differential equation

$$\frac{d\varphi(x)}{dx} = -\frac{i}{\hbar v} V(x)\varphi(x)$$

with the boundary condition

$$\varphi(-\infty) = 1,$$

and the solution

$$\psi(x) = e^{ikx - \frac{i}{\hbar v}\int_{-\infty}^{x} V(x')dx'}. \tag{55}$$

The restrictions underlying this result may be clearly seen from the above. We require that both $V(x)$ and $\varphi(x)$ vary slowly within a wavelength. The first of these conditions is $ka \gg 1$ where a is the width of the potential. The second condition, is indicated by the form derived for $\varphi(x)$. We evidently require

$$k \gg \frac{V}{\hbar v}$$

or

$$1 \gg V/E\ .$$

These are the conditions, Eq. (50), stated earlier.

It should be noted particularly that even though the assumptions

$$ka \gg 1$$

and

$$V/E \ll 1$$

are required above, no restriction has been placed on the product of these two quantities. Now their product is

$$ka \cdot \frac{V}{E} = 2\,\frac{Va}{\hbar v}\ , \tag{62}$$

so we see that the present approximation, in contrast with those discussed earlier, remains valid for arbitrary values of the important parameter

$$Va/\hbar v.$$

Before discussing higher approximations we might point out that the form Eq. (55) for the wave function may also be reached by means of the W. K. B. method. One has only to expand the familiar W. K. B. approximation to a one-dimensional wave function in power of V. But, unfortunately this is a shortcut confined to one dimensional problems. The one-dimensional W. K. B. approximation, as we have noted earlier, is unusual in that it need not require

$$Va/\hbar v \gg 1\ ,$$

and may in this case overlap the present approximation. In two or more dimensions, however, this overlap disappears. The direct generalization of the method we are discussing yields results which only coincide with those of the W. K. B. approximation in the limit

$$Va/\hbar v \longrightarrow \infty\ .$$

In order to improve the accuracy of the approximation, explicit account must be taken of the back-scattered wave. For this purpose, we write the wave function as,

$$\psi(x) = e^{ikx}\varphi_+(x) + e^{-ikx}\varphi_-(x)\,. \tag{63}$$