The Joint Institute for Nuclear Astrophysics

JINA "Methods of Direct Nuclear Reactions" School

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Session (learning) aims:

To calculate approximate solutions of the Schrodinger equation for states of **two, three** or more bodies at ‘high energies’ by using the eikonal (forward scattering) approximation in the reaction dynamics.

To bring out the importance of the eikonal S-matrix, a function if the impact parameter of the projectile or component of the projectile, to this formulation of the reaction and scattering of the interacting systems.

Through hands on exercises, to gain a feel for how accurate the eikonal approximation is as a function of the projectile energy and to calculate both simple and composite projectile elastic scattering using the method.

Session (learning) outcomes:

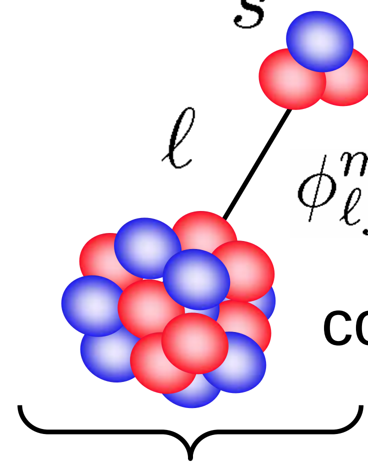
To gain familiarity in using a small suite of programs for approximate solutions of the Schrodinger equation and for elastic scattering using the eikonal approximation. To calculate the eikonal S-matrix from underlying two-body potentials for both inert and for composite projectiles.

To understand the important role of the eikonal S-matrix in this formulation of the reaction and scattering of both inert and of composite nuclear systems.

To evaluate the eikonal phase shift functions analytically in simple cases and numerically in the case of a general spin-independent potential energy of interaction. To compare these and the resulting elastic scattering cross sections with exact calculations and be able to combine two-body S-matrices to predict elastic scattering of a three-body system.

Our … bits and pieces…

$$U(r) = \underbrace{V_C(r)}_{\text{Coulomb}} + \underbrace{V(r) + V_{so}(r)\vec{\ell}\cdot\vec{s}}_{\text{Nuclear}}$$



Need descriptions of wave functions of:

(1) Bound states of nucleons or clusters (valence particles) to a core (that is assumed for now to have spin zero).

(2) Unbound scattering or resonant states at low energy

(3) Distorted waves of such bodies in complex potentials

$$U(r) = V_C(r) + V(r) + iW(r) + V_{so}(r)\vec{\ell}\cdot\vec{s}$$

Optical potentials - parameterisations

$$U(r) = V_C(r) + V(r) + iW(r) + V_{so}(r)\vec{\ell}\cdot\vec{s}$$

$$V(r) = -\frac{V_R}{[1+\exp(X_R)]}\ , \qquad X_i = \frac{r-R_i}{a_i}$$

$$V_{so}(r) = -\frac{2V_{so}}{r\,a_{so}}\frac{\exp(X_{so})}{[1+\exp(X_{so})]^2}\ ,$$

$$W(r) = -\frac{W_V}{[1+\exp(X_V)]} - \frac{4W_S\exp(X_S)}{[1+\exp(X_S)]^2}\ ,$$

$$R_i = r_i\,A_2^{1/3} \quad \text{or} \quad R_i = r_i\left[A_1^{1/3} + A_2^{1/3}\right]$$

Phase shift and partial wave S-matrix: Recall

$$u_{k\ell j}(r) \quad \rightarrow \quad e^{i\delta_{\ell j}}\underbrace{[\cos\delta_{\ell j}F_\ell(\eta,kr)+\sin\delta_{\ell j}G_\ell(\eta,kr)]}$$

If *U(r)* is real, the phase shifts $\delta_{\ell j}$ are real, and […] also

$$u_{k\ell j}(r) \quad \rightarrow \quad (i/2)[\underline{H_\ell^{(-)}}(\eta,kr)-S_{\ell j}\underline{H_\ell^{(+)}}(\eta,kr)]$$

$S_{\ell j}=e^{2i\delta_{\ell j}}$

Ingoing waves

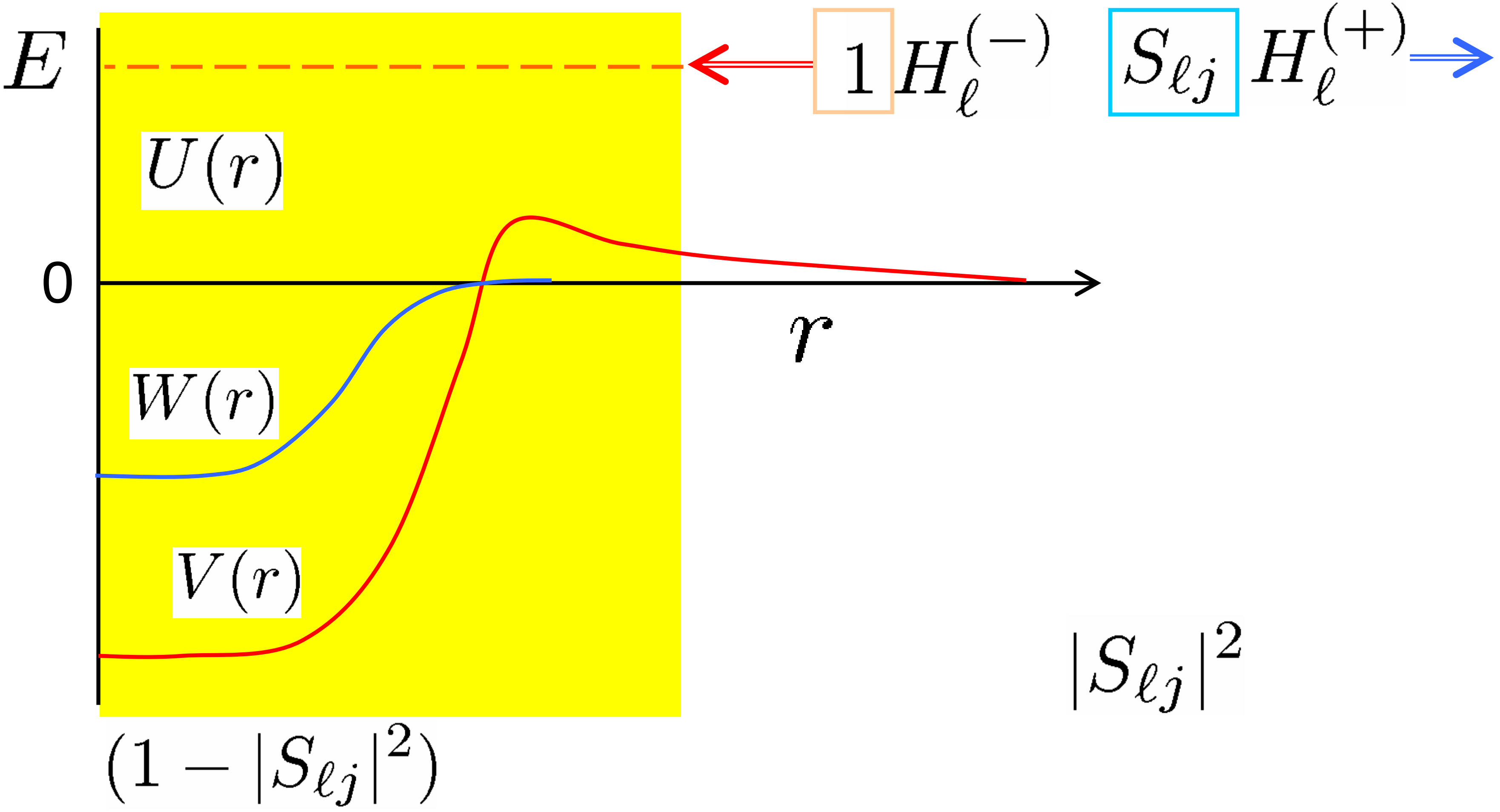
outgoing waves

$|S_{\ell j}|^2$ survival probability in the scattering

$(1-|S_{\ell j}|^2)$ absorption probability in the scattering

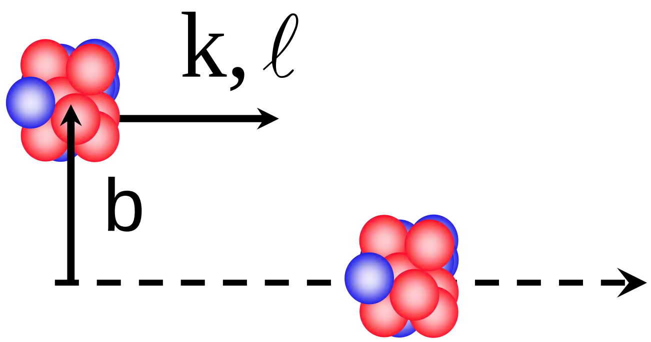
Ingoing and outgoing waves amplitudes

$$u_{k\ell j}(r) \rightarrow (i/2)\left[1\, H_\ell^{(-)} - S_{\ell j} H_\ell^{(+)}\right]$$



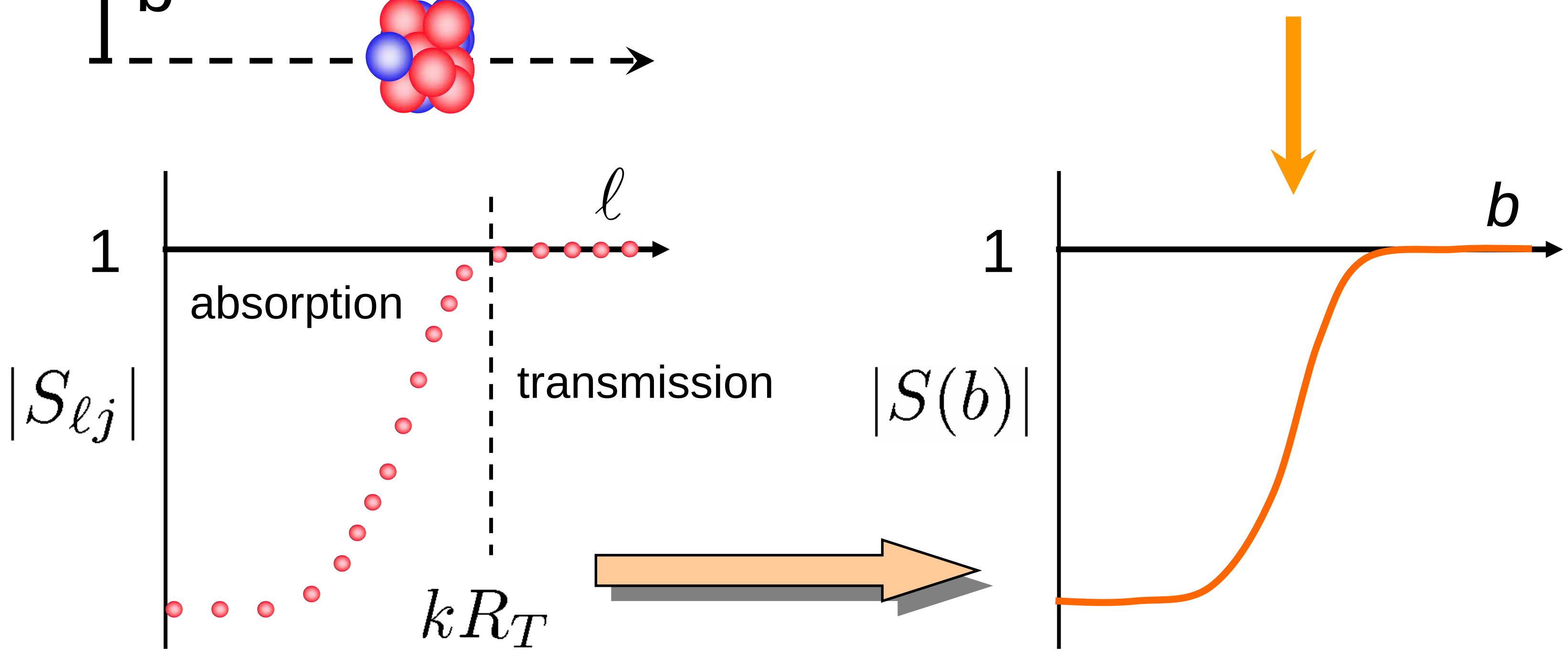
Semi-classical models for the S-matrix - S(b)

b=impact parameter



for high energy/or large mass, semi-classical ideas are good

$kb \cong \ell$, actually $\Rightarrow \ell$+1/2



Use an old friend – and meet some new ones

bound (bound states solver – see **bound.outline**)

eikonal_s (for eikonal S-matrix from a specified interaction potential - **eikonal_s.outline**)

glauber (elastic scattering calculation from a specifies eikonal S-matrix - **glauber.outline**)

knockout (composite two-body projectile S-matrix from a bound state wave function and component S-matrices – **knockout.outline**)

Eikonal approximation: point particles (1)

Approximate (semi-classical) scattering solution of

$$\left(-\frac{\hbar^2}{2\mu}\nabla_r^2 + U(r) - E_{cm}\right)\chi_{\vec{k}}^{+}(\vec{r}) = 0, \qquad \mu = \frac{m_c m_v}{m_c + m_v}$$

$$\left(\nabla_r^2 - \frac{2\mu}{\hbar^2}U(r) + k^2\right)\chi_{\vec{k}}^{+}(\vec{r}) = 0$$

valid when $|U|/E \ll 1, \quad ka \gg 1$ → small wavelength high energy

Key steps are: (1) the distorted wave function is written

$$\chi_{\vec{k}}^{+}(\vec{r}) = \exp(i\vec{k}\cdot\vec{r})\,\omega(\vec{r})$$

all effects due to *U(r)*, modulation function

(2) Substituting this product form in the Schrodinger Eq.

$$\left[2i\vec{k}\cdot\nabla\omega(\vec{r}) - \frac{2\mu}{\hbar^2}U(r)\omega(\vec{r}) + \nabla^2\omega(\vec{r})\right]\exp(i\vec{k}\cdot\vec{r}) = 0$$

Eikonal approximation: point neutral particles (2)

$$\left[2i\vec{k}\cdot\nabla\omega(\vec{r}) - \frac{2\mu}{\hbar^2}U(r)\omega(\vec{r}) + \nabla^2\omega(\vec{r})\right]\exp(i\vec{k}\cdot\vec{r}) = 0$$

The conditions $|U|/E \ll 1, \quad ka \gg 1$ → imply that

$$2\vec{k}\cdot\nabla\omega(\vec{r}) \gg \nabla^2\omega(\vec{r})$$

Slow spatial variation cf. k

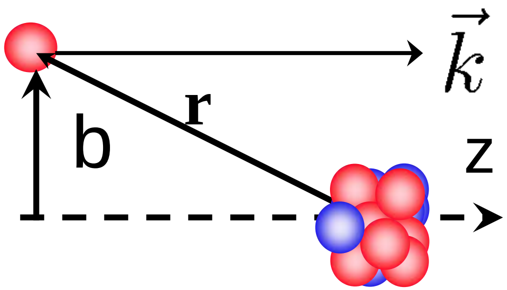
and choosing the z-axis in the beam direction $\vec{k}$

$$\frac{d\omega}{dz} \approx -\frac{i\mu}{\hbar^2 k}U(r)\omega(\vec{r})$$

with solution

phase that develops with z

$$\omega(\vec{r}) = \exp\left[-\frac{i\mu}{\hbar^2 k}\int_{-\infty}^{z} U(r)dz'\right]$$



1D integral over a straight line path through U at the impact parameter b

Eikonal approximation: point neutral particles (3)

$$\chi^{+}_{\vec{k}}(\vec{r}) = \exp(i\vec{k}\cdot\vec{r})\ \omega(\vec{r}) \approx \exp(i\vec{k}\cdot\vec{r})\ \exp\left[-\frac{i\mu}{\hbar^2 k}\int_{-\infty}^{z} U(r)dz'\right]$$

So, after the interaction and as $z \to \infty$

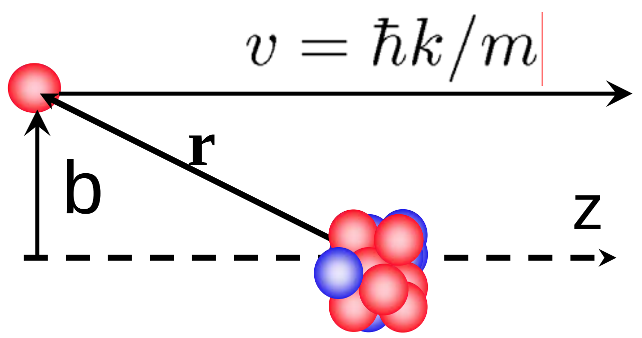
$$\chi^{+}_{\vec{k}}(\vec{r}) \to \exp(i\vec{k}\cdot\vec{r})\ \exp\left[-\frac{i\mu}{\hbar^2 k}\int_{-\infty}^{\infty} U(r)dz'\right] = S(b)\exp(i\vec{k}\cdot\vec{r})$$

$$\chi^{+}_{\vec{k}}(\vec{r}) \to S(b)\exp(i\vec{k}\cdot\vec{r})$$

S(b) is amplitude of the forward going outgoing waves from the scattering at impact parameter b

Eikonal approximation to the S-matrix S(b)

$$S(b) = \exp\left[-\frac{i}{\hbar v}\int_{-\infty}^{\infty} U(r)dz'\right]$$

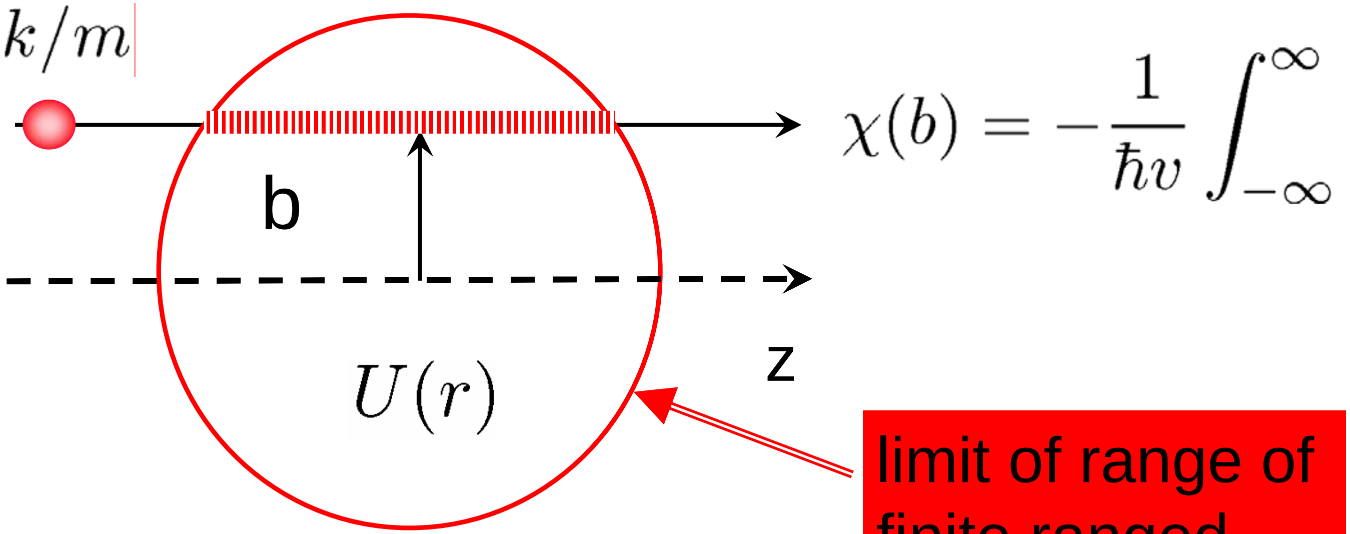


Moreover, the structure of the theory generalises simply to few-body projectiles

Eikonal approximation: point particles (summary)

$$\chi^{+}_{\vec{k}}(\vec{r}) = \exp(i\vec{k}\cdot\vec{r})\ \exp\left[-\frac{i\mu}{\hbar^2 k}\int_{-\infty}^{z} U(r)dz'\right]$$

$$v = \hbar k/m$$



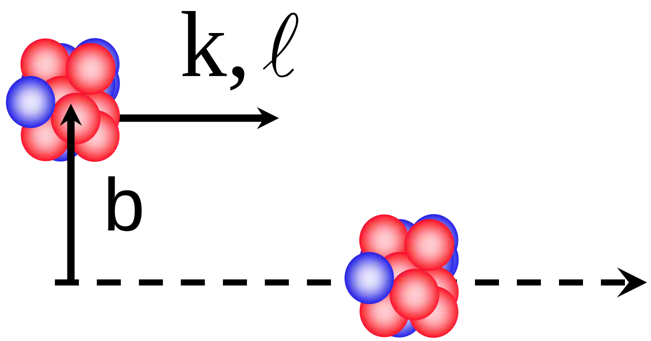
$$\chi(b) = -\frac{1}{\hbar v}\int_{-\infty}^{\infty} U(r)dz$$

$$\chi^{+}_{\vec{k}}(\vec{r}) \rightarrow S(b)\exp(i\vec{k}\cdot\vec{r})$$

$$S(b) = \exp\left[i\chi(b)\right] = \exp\left[-\frac{i}{\hbar v}\int_{-\infty}^{\infty} U(r)dz'\right]$$

Semi-classical models for the S-matrix - S(b)

b=impact parameter



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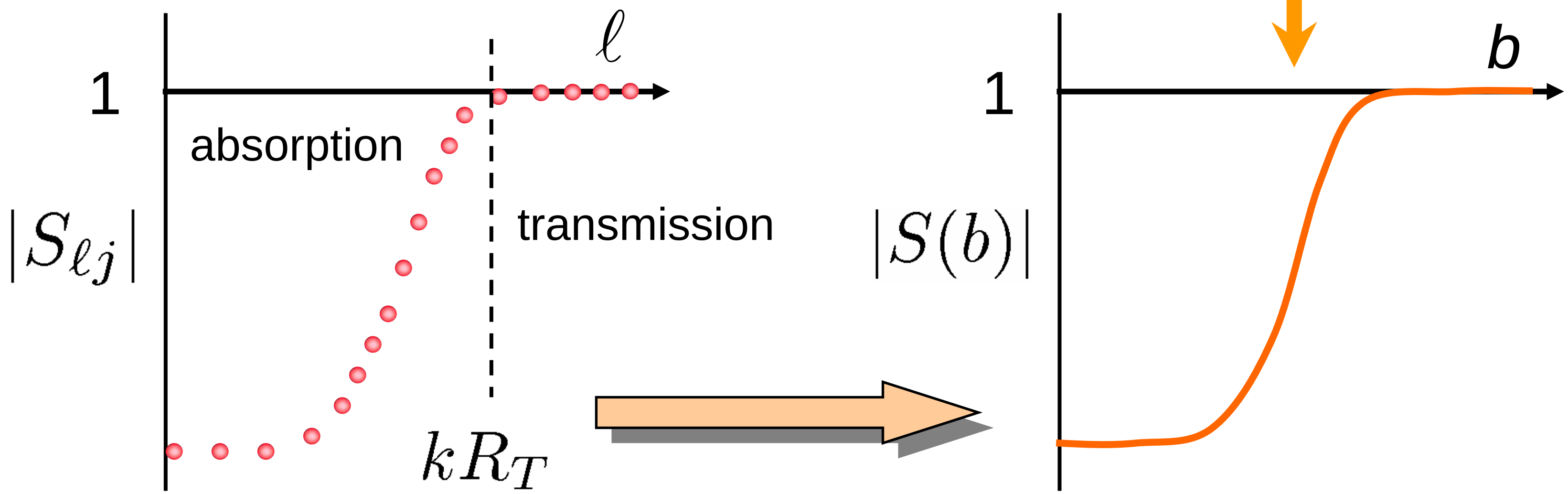
$kb \cong \ell$, actually $\Rightarrow \ell + 1/2$



$$u_{k\ell j}(r) \to (i/2)[H_\ell^{(-)} - S_{\ell j} H_\ell^{(+)}]$$

$$S(b) = \exp\left[-\frac{i}{\hbar v}\int_{-\infty}^{\infty} U(r)\,dz'\right]$$

Point particle scattering – cross sections

All cross sections, etc. can be computed from the S-matrix, in either the partial wave or the eikonal (impact parameter) representation, for example (spinless case):

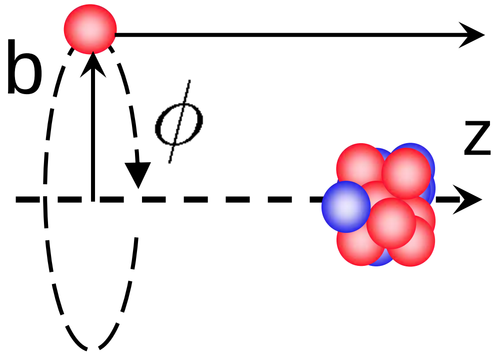
$$\sigma_{el} = \frac{\pi}{k^2}\sum_{\ell=0}^{\infty}(2\ell+1)|1-S_\ell|^2 \approx \int d^2\vec{b}\; |1-S(b)|^2$$

$$\sigma_R = \frac{\pi}{k^2}\sum_{\ell=0}^{\infty}(2\ell+1)(1-|S_\ell|^2) \approx \int d^2\vec{b}\; (1-|S(b)|^2)$$

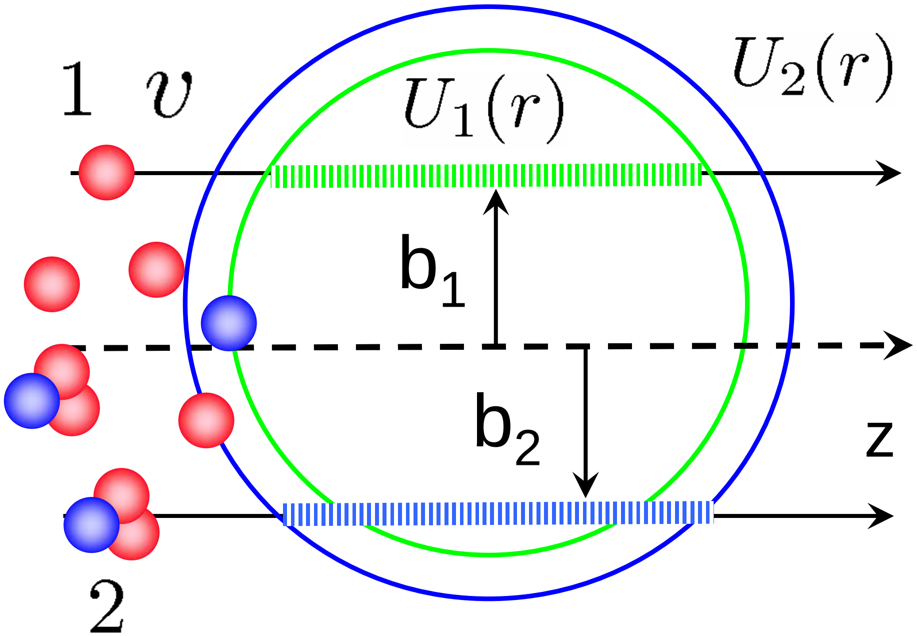
$$\sigma_{tot} = \sigma_{el} + \sigma_R = 2\int d^2\vec{b}\; [1-\mathrm{Re}.S(b)]$$ etc.

and where (cylindrical coordinates)

$$\int d^2\vec{b} \equiv \int_0^\infty b\,db \int_0^{2\pi} d\phi = 2\pi \int_0^\infty b\,db$$



Eikonal approximation: several particles (preview)



$$\chi_i(b) = -\frac{1}{\hbar v}\int_{-\infty}^{\infty} U_i(r)dz$$

Total interaction energy

$$U(r_1,\ldots) = \sum_i U_i(r_i)$$

$$S_i(b_i) = \exp\left[i\chi_i(b_i)\right] = \exp\left[-\frac{i}{\hbar v}\int_{-\infty}^{\infty} U_i(r_i)dz'\right]$$

$$\chi(b_1,\ldots) = -\frac{1}{\hbar v}\int_{-\infty}^{\infty}\sum_i U_i(r_i)dz$$

with composite objects we will get products of the S-matrices

$$\exp[i\chi(b_1,\ldots)] = \prod_i S_i(b_i)$$

Point particle – the differential cross section

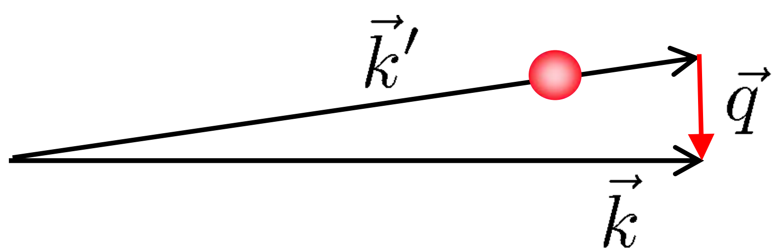
Using the standard result from scattering theory, the elastic scattering amplitude is

$$f(\theta) = -\frac{\mu}{2\pi\hbar^2}\int d\vec{r}\,\exp(-i\vec{k}'\cdot\vec{r})\,U(r)\,\chi^{+}_{\vec{k}}(\vec{r})$$

$$= -\frac{\mu}{2\pi\hbar^2}\int d\vec{r}\,\exp(-i\vec{k}'\cdot\vec{r})\,U(r)\,\boxed{\exp(i\vec{k}\cdot\vec{r})\,\omega(\vec{r})}$$

$$= -\frac{\mu}{2\pi\hbar^2}\int d\vec{r}\,\exp(i\vec{q}\cdot\vec{r})\,U(r)\,\omega(\vec{r})$$

with $\vec{q} = \vec{k} - \vec{k}',\ q = 2k\sin(\theta/2)$ is the momentum transfer. Consistent with the earlier high energy (forward scattering) approximation



$$\vec{q}\cdot\vec{r} \approx \vec{q}\cdot\vec{b}$$

$$\vec{q}\cdot\vec{k} \approx 0$$

Point particles – the differential cross section

So, the elastic scattering amplitude

$$f(\theta) = -\frac{\mu}{2\pi\hbar^2}\int d\vec{r}\,\exp(i\vec{q}\cdot\vec{r})\,U(r)\,\omega(\vec{r})$$

$$\begin{cases} \dfrac{d\omega}{dz} = -\dfrac{i\mu}{\hbar^2 k}U(r)\omega(\vec{r}) \\ U(r)\omega(\vec{r}) = \dfrac{i\hbar^2 k}{\mu}\dfrac{d\omega}{dz} \end{cases}$$

is approximated by

$$f_{eik}(\theta) = -\frac{ik}{2\pi}\int d^2\vec{b}\,\exp(i\vec{q}\cdot\vec{b})\int_{-\infty}^{\infty}\frac{d\omega}{dz}$$

Performing the z- and azimuthal ϕ integrals

$$f_{eik}(\theta) = -ik\int_0^{\infty} b\,db\,J_0(qb)\,[S(b)-1]$$

$J_0(qb)$ Bessel function

$$S(b) = \exp[i\chi(b)] = \exp\left[-\frac{i}{\hbar v}\int_{-\infty}^{\infty}U(r)dz'\right]$$

Point particle – the Coulomb interaction

Treatment of the Coulomb interaction (as in partial wave analysis) requires a little care. Problem is, eikonal phase integral due to Coulomb potential diverges logarithmically.

$$\chi_C(b) = -\frac{1}{\hbar v}\int_{-a}^{+a} V_C(r)\,dz$$

Must 'screen' the potential at some large screening radius

$$f_{eik}(\theta) = e^{i\chi_a}\left[f_{pt}(\theta) - ik\int_0^\infty b\,db\,J_0(qb)\,e^{i\chi_{pt}}[\bar{S}(b)-1]\right]$$

overall unobservable screening phase

usual Coulomb (Rutherford) point charge amplitude

nuclear scattering in the presence of Coulomb

$$\bar{\chi}(b) = \chi_N(b) + \chi_\rho(b) - \chi_{pt}(b)$$

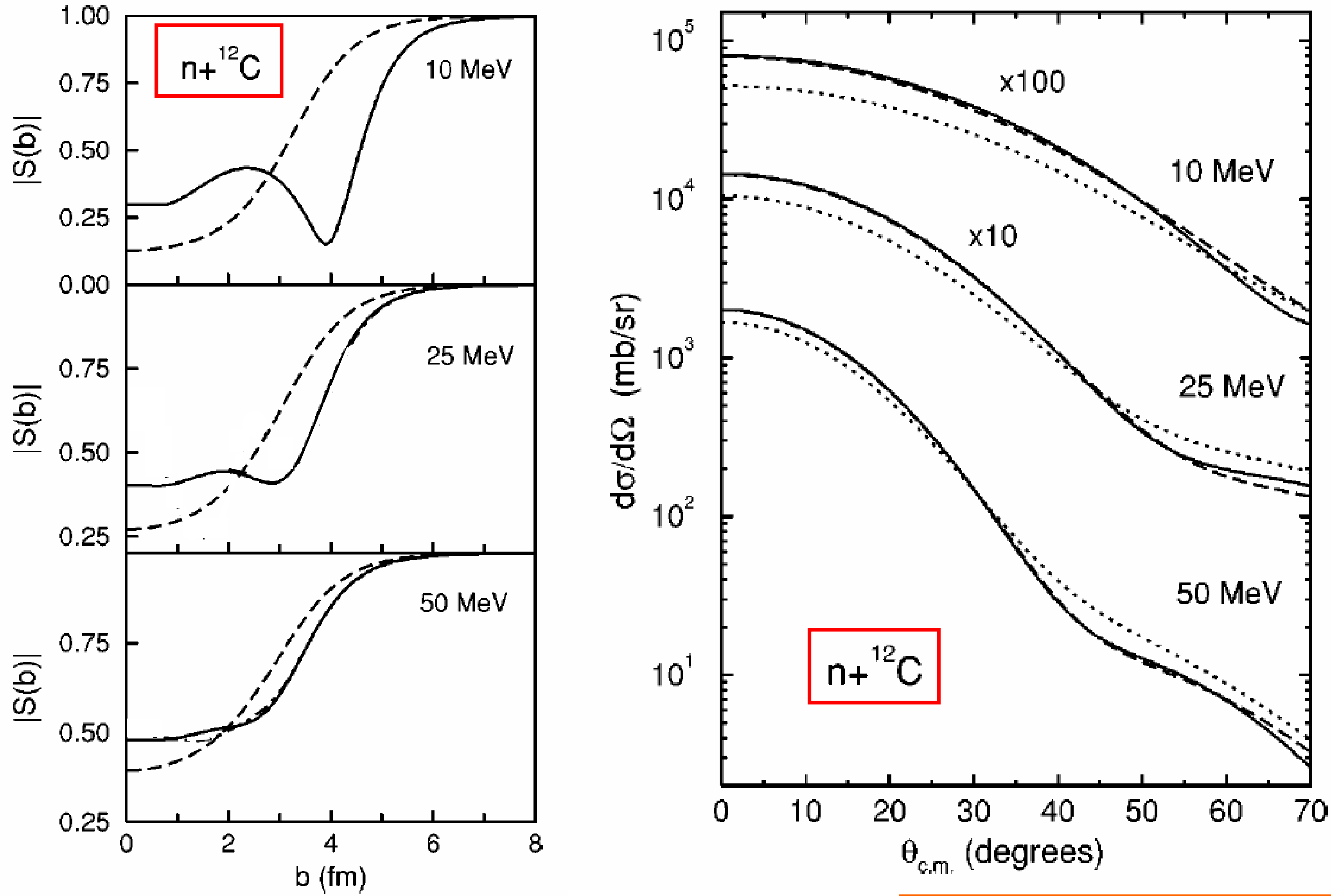
nuclear phase

Due to finite charge distribution

$$\chi_{pt}(b) = 2\eta\ln(kb)$$

See e.g. J.M. Brooke, J.S. Al-Khalili, and J.A. Tostevin PRC **59** 1560

Accuracy of the eikonal S(b) and cross sections



J.M. Brooke, J.S. Al-Khalili, and J.A. Tostevin PRC **59** 1560

do 100 MeV/nucleon

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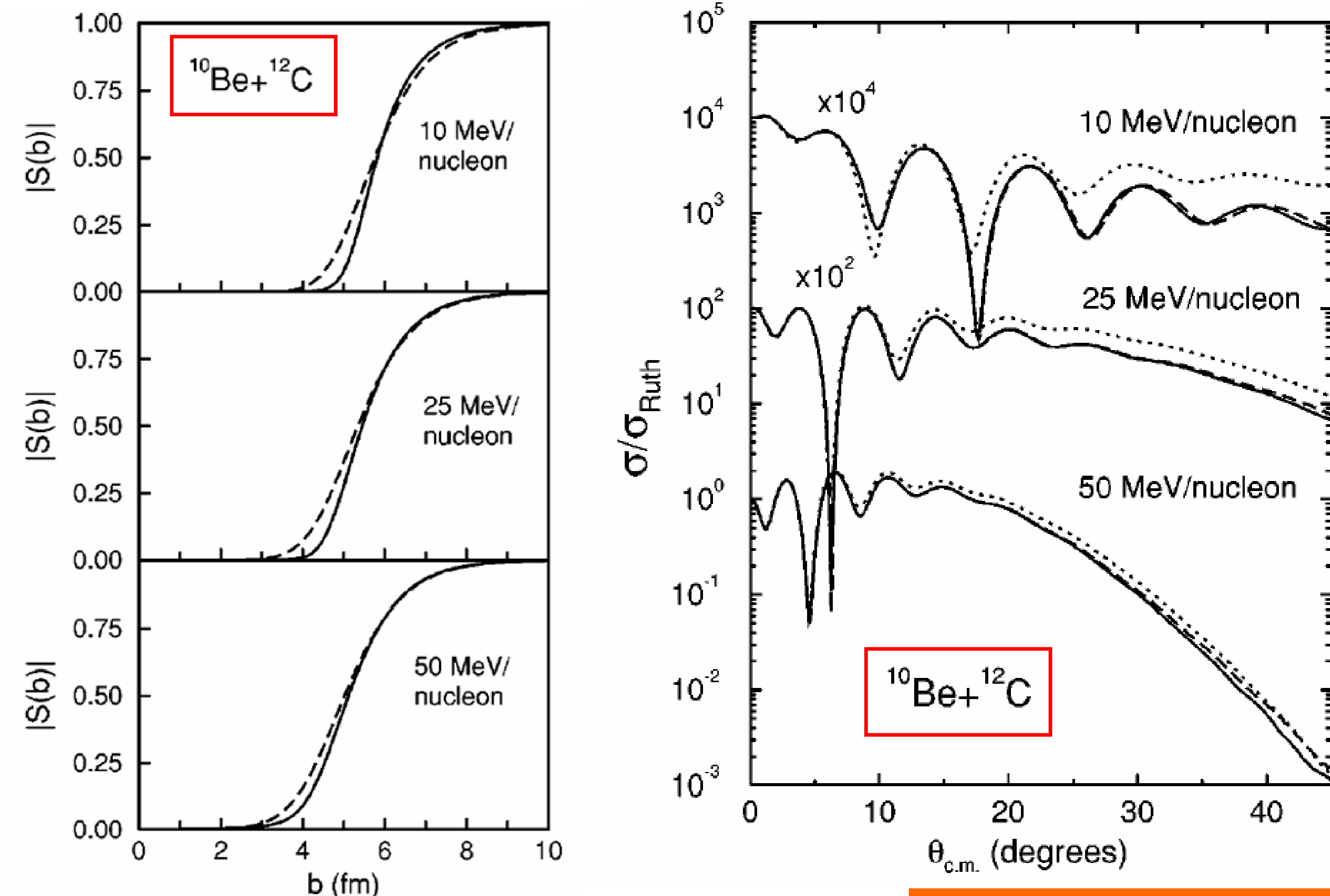

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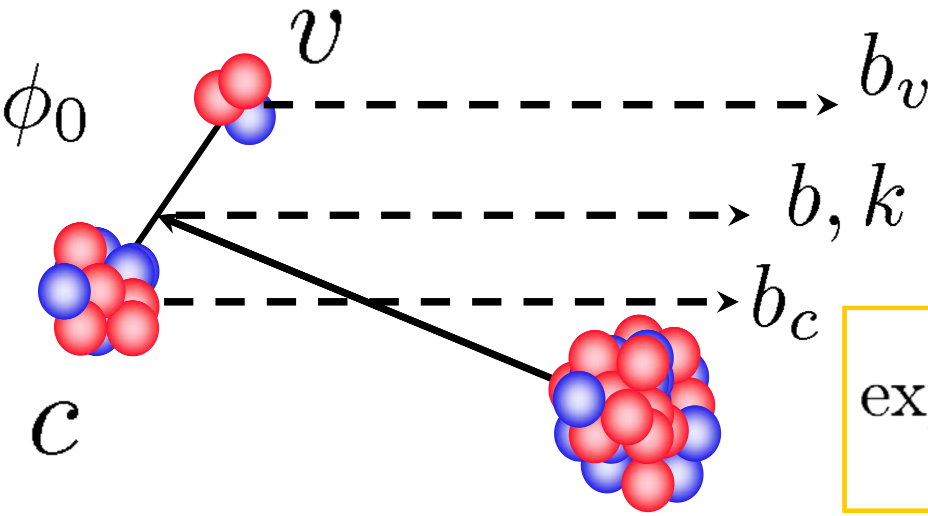
Eikonal approach – generalisation to composites



Total interaction energy

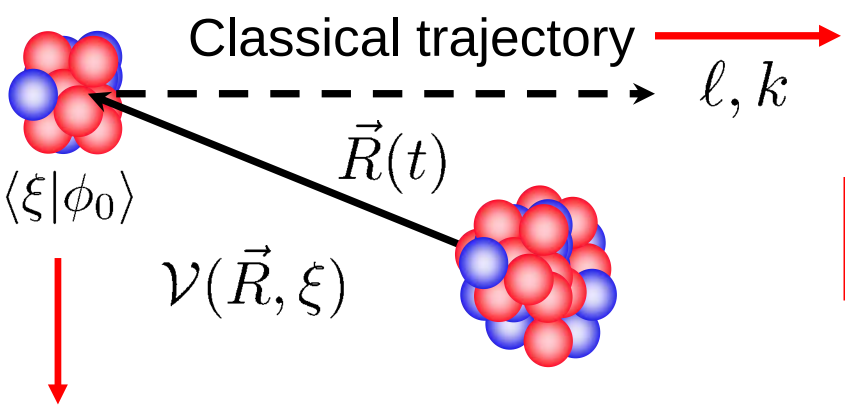
$$U(r_1, \ldots) = \sum_{i=c,v} U_i(r_i)$$

$$\exp[i\chi(b_1, \ldots)] = \prod_{i=c,v} S_i(b_i)$$

$$S_p(b) = \langle \phi_0 | S_c(b_c) S_v(b_v) | \phi_0 \rangle$$

You can now calculate bound states (**`bound`**) and eikonal S-matrices (**`eikonal_s`**) and can calculate this composite S-matrix (using **`knockout`**). The elastic scattering of c, v or the composite can then be calculated (using **`glauber`**). So you can now calculate the elastic scattering of the neutron, ^{10}Be, and the composite halo system ^{11}Be ?

Semi-classical – e.g. Alder and Winther theory



$$V(R) = \langle \phi_0 | \mathcal{V}(\vec{R}, \xi) | \phi_0 \rangle$$

$$H = H_0(\xi) + \mathcal{V}(\vec{R}, \xi)$$

$$H_0(\xi)|\phi_\alpha\rangle = \mathcal{E}_\alpha |\phi_\alpha\rangle$$

$$a_\alpha(\ell, t \to -\infty) = \delta_{\alpha 0}$$

$$\left[H_0(\xi) + \mathcal{V}(\vec{R}(t), \xi)\right] \psi_\ell(\xi, t) = i\hbar \frac{\partial \psi_\ell(\xi, t)}{\partial t}$$

$$\psi_\ell(\xi, t) = \sum_\alpha a_\alpha(\ell, t)\, \phi_\alpha(\xi) \exp(-i\mathcal{E}_\alpha t/\hbar)$$

$$i\hbar \dot{a}_\alpha(\ell, t) = \sum_\beta \langle \phi_\alpha | \mathcal{V}(\vec{R}, \xi) | \phi_\beta \rangle \exp(i[\mathcal{E}_\alpha - \mathcal{E}_\beta]/\hbar)\ a_\beta(\ell, t)$$

Homework: to become familiar with these ideas

`bound` (bound states solver – see **`bound.outline`**)

`eikonal_s` (for eikonal S-matrix from a specified interaction potential - **`eikonal_s.outline`**)

`glauber` (elastic scattering calculation from a specifies eikonal S-matrix - **`glauber.outline`**)

`knockout` (composite two-body projectile S-matrix from a bound state wave function and component S-matrices – **`knockout.outline`**)