TALENT Course 6: Theory for exploring nuclear reaction experiments
Outline project proposal

Project name: Non-Local Potentials in the Eikonal Approximation

Researcher(s): Luke Titus
Affiliation: Michigan State University
Supervisor(s): Filomena Nunes

Project outline and aims:
The goal of this project is to implement the Eikonal approximation for elastic scattering when non-local potentials are considered. As a first approximation, the Frahn-Lemmer form for the non-local potential will be used. This potential was later refined, and a partial wave decomposition was derived by Perey and Buck.

Methodology:
The Eikonal Approximation: Local Potential

$$\left(-\frac{\hbar^2}{2\mu}\nabla_r^2 + U(\mathbf{r}) - E\right)\Psi(\mathbf{r}) = 0 \tag{1}$$

Assume

$$\Psi(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}}w(\mathbf{r}) \tag{2}$$

Thus

$$\left(2i\mathbf{k}\cdot\nabla w(\mathbf{r}) - \frac{2\mu}{\hbar^2}U(\mathbf{r})w(\mathbf{r}) + \nabla^2 w(\mathbf{r})\right)e^{i\mathbf{k}\cdot\mathbf{r}} = 0 \tag{3}$$

Neglecting $\nabla^2 w(\mathbf{r})$ and assuming that $\mathbf{k} = k\hat{z}$ so that $\mathbf{k}\cdot\nabla w(\mathbf{r}) = k\frac{dw}{dz}$

$$\left(ik\frac{dw}{dz} - \frac{\mu}{\hbar^2}U(\mathbf{r})w(\mathbf{r})\right)e^{i\mathbf{k}\cdot\mathbf{r}} = 0 \tag{4}$$

The above equation will always equal zero only when what is inside the parentheses always equals zero, thus

$$\frac{dw}{dz} = -\frac{i\mu}{\hbar^2 k}U(\mathbf{r})w(\mathbf{r}) \tag{5}$$

Dividing by $w(\mathbf{r})$, multiplying by dz, integrating, then exponentiating the result

$$w(b,z) = \exp\left[-\frac{i\mu}{\hbar^2 k}\int_{-\infty}^{z} U(b,z')dz'\right] \tag{6}$$

The Eikonal Approximation: Non-Local Potential
Making the same approximation for $\Psi(\mathbf{r})$, but assuming a non-local potential

$$\left(2i\mathbf{k}\cdot\nabla w(\mathbf{r}) - \frac{2\mu}{\hbar^2}\frac{\int d\mathbf{r}' U(\mathbf{r},\mathbf{r}')w(\mathbf{r}')e^{i\mathbf{k}\cdot\mathbf{r}'}}{e^{i\mathbf{k}\cdot\mathbf{r}}} + \nabla^2 w(\mathbf{r})\right)e^{i\mathbf{k}\cdot\mathbf{r}} = 0 \tag{7}$$

The integral over $d\mathbf{r}'$ is over all space, so $\mathbf{r}' = (b', z')$ where $b' = \sqrt{x'^2 + y'^2}$. Multiplying the integral term by $e^{i\mathbf{k}\cdot\mathbf{r}}e^{-i\mathbf{k}\cdot\mathbf{r}}$, neglecting the ∇^2 term, and treating $\mathbf{k}\cdot\nabla w(\mathbf{r})$ as we did before.

$$\left(ik\frac{dw}{dz} - \frac{\mu e^{-i\mathbf{k}\cdot\mathbf{r}}}{\hbar^2}\int d\mathbf{r}' U(\mathbf{r},\mathbf{r}')w(\mathbf{r}')e^{i\mathbf{k}\cdot\mathbf{r}'}\right)e^{i\mathbf{k}\cdot\mathbf{r}} = 0 \tag{8}$$

Multiplying the top and bottom of the integral term by $w(\mathbf{r})$

$$\left(ik\frac{dw}{dz} - \frac{\mu e^{-i\mathbf{k}\cdot\mathbf{r}}}{\hbar^2}\frac{\int d\mathbf{r}' U(\mathbf{r},\mathbf{r}')w(\mathbf{r}')e^{i\mathbf{k}\cdot\mathbf{r}'}}{w(\mathbf{r})}w(\mathbf{r}) \right) e^{i\mathbf{k}\cdot\mathbf{r}} = 0 \tag{9}$$

So that

$$\left(ik\frac{dw}{dz} - \frac{\mu}{\hbar^2}\mathcal{U}(\mathbf{r})w(\mathbf{r}) \right) e^{i\mathbf{k}\cdot\mathbf{r}} = 0 \tag{10}$$

With

$$\mathcal{U}(\mathbf{r}) = \frac{e^{-i\mathbf{k}\cdot\mathbf{r}}\int d\mathbf{r}' U(\mathbf{r},\mathbf{r}')w(\mathbf{r}')e^{i\mathbf{k}\cdot\mathbf{r}'}}{w(\mathbf{r})} \tag{11}$$

At this point we can formally solve for $w(\mathbf{r}) = w(b,z)$ in the same way we did with a local potential. This results in

$$w(b,z) = \exp\left[-\frac{i\mu}{\hbar^2 k}\int_{-\infty}^{z} \mathcal{U}(\mathbf{r}'')dz'' \right] \tag{12}$$

The notation being used here is that $\mathbf{r}'' = (b,z'')$, with $b = \sqrt{x^2+y^2}$. Unfortunately, solving this equation is not so straight forward since $w(\mathbf{r})$ appears on both sides of the equation. Restoring all the factors to make this more explicit, we have

$$w(b,z) = \exp\left[-\frac{i\mu}{\hbar^2 k}\int_{-\infty}^{z} \frac{e^{i\mathbf{k}\cdot\mathbf{r}''}}{w(\mathbf{r}'')}\left[\int d\mathbf{r}' U(\mathbf{r}'',\mathbf{r}')w(\mathbf{r}')e^{i\mathbf{k}\cdot\mathbf{r}'} \right] dz'' \right] \tag{13}$$

The way to go about solving this equation is through iteration. We will first evaluate the local version of $w(\mathbf{r})$ by using (6) with some suitable local equivalent potential. Calling this function $w_o(\mathbf{r})$ and plugging it into the right hand side of the above equation, we can calculate the first iteration of the non-local $w(\mathbf{r})$. In full, with n labeling the step in the iteration process, we have

$$w_n(b,z) = \exp\left[-\frac{i\mu}{\hbar^2 k}\int_{-\infty}^{z} \frac{e^{i\mathbf{k}\cdot\mathbf{r}''}}{w_{n-1}(\mathbf{r}'')}\left[\int d\mathbf{r}' U(\mathbf{r}'',\mathbf{r}')w_{n-1}(\mathbf{r}')e^{i\mathbf{k}\cdot\mathbf{r}'} \right] dz'' \right] \tag{14}$$

The iteration scheme proceeds until $w_n(\mathbf{r})$ agrees with $w_{n-1}(\mathbf{r})$ within a desired level of accuracy. Of course, certain technical aspects need to be worked out in order to evaluate the nasty expression above.

Key references:

1. W.E. Frahn and R.H. Lemmer, Nuovo Cim. 5 (1957) 1564
2. F. Perey and B. Buck, Nucl. Phys. 32 (1962) 353