

R-matrix Analysis (I)

TALENT Course 6
Theory for exploring nuclear reaction experiments

GANIL 1st-19th July

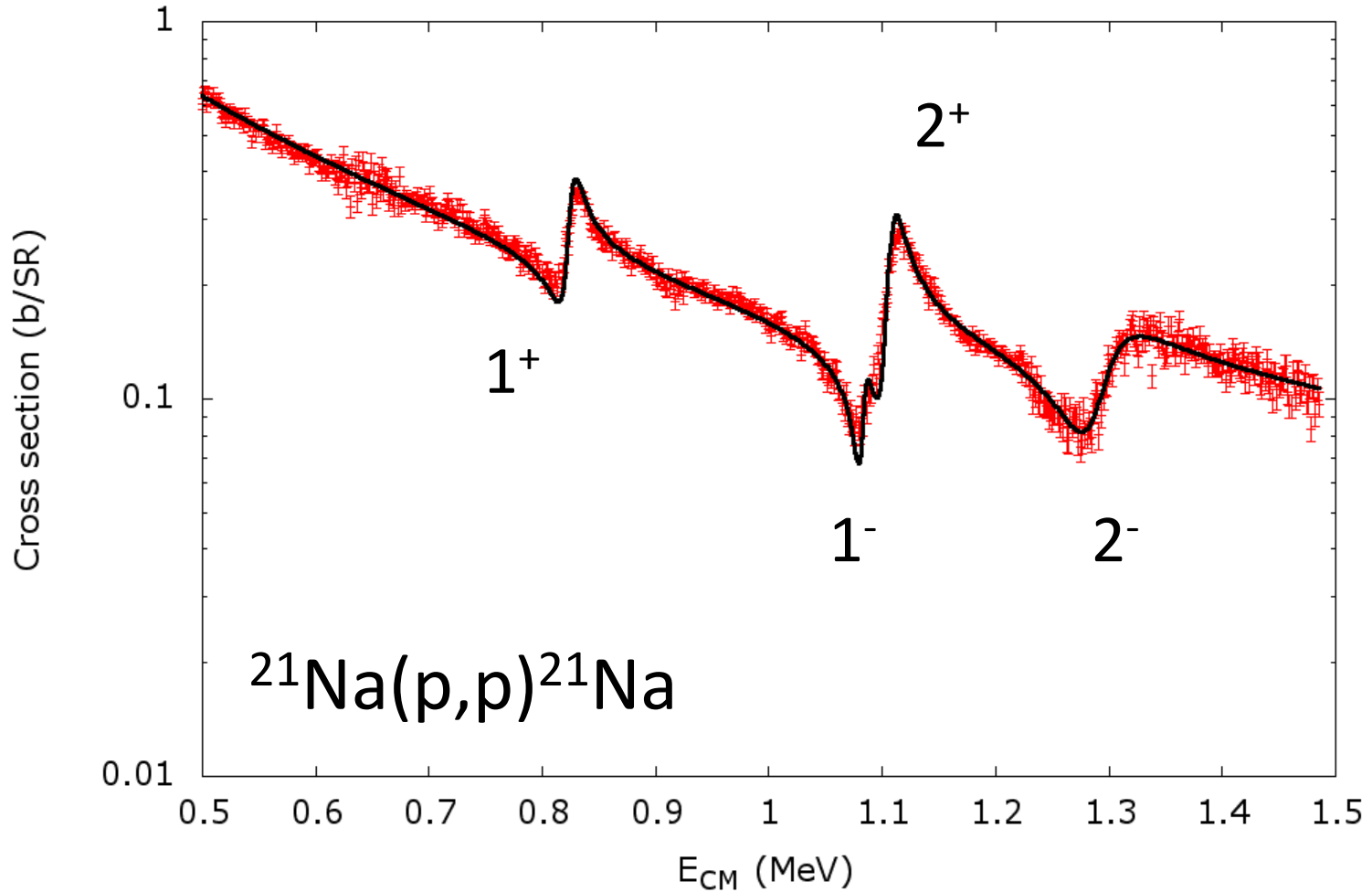
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Introduction

Why do we need R-matrix phenomenology?



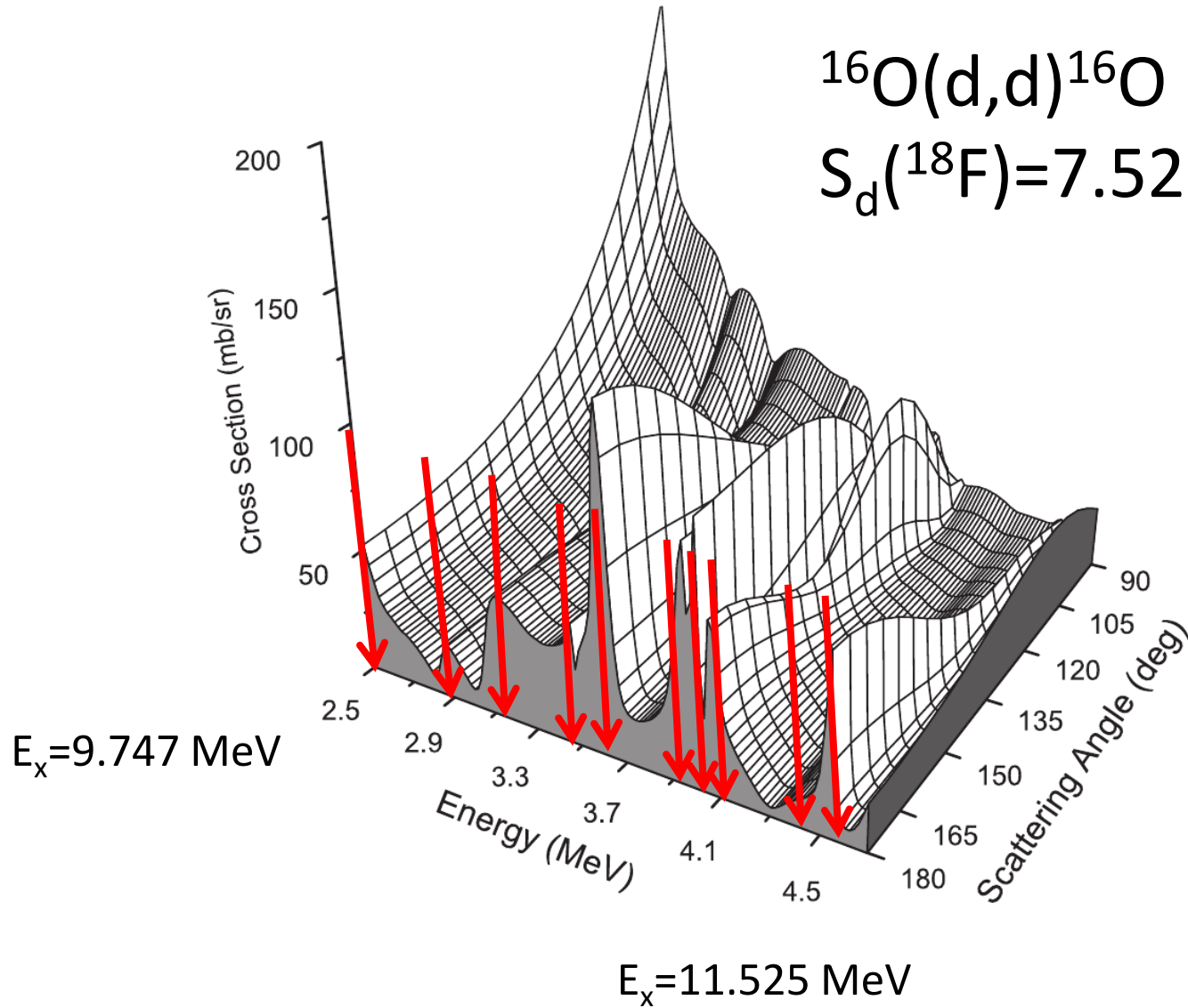
Determining nuclear structure



Materials analysis

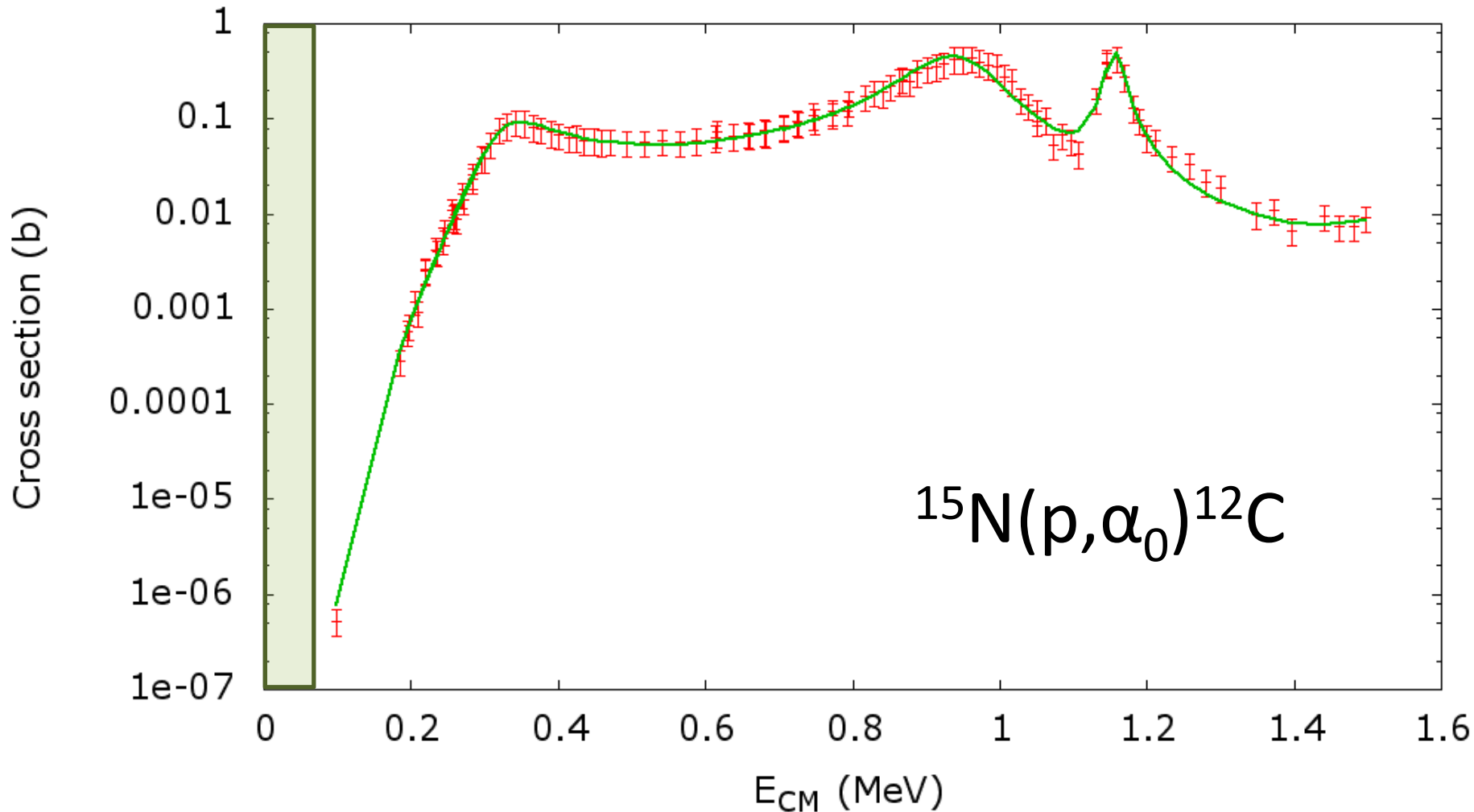


$$S_d(^{18}\text{F})=7.525 \text{ MeV}$$

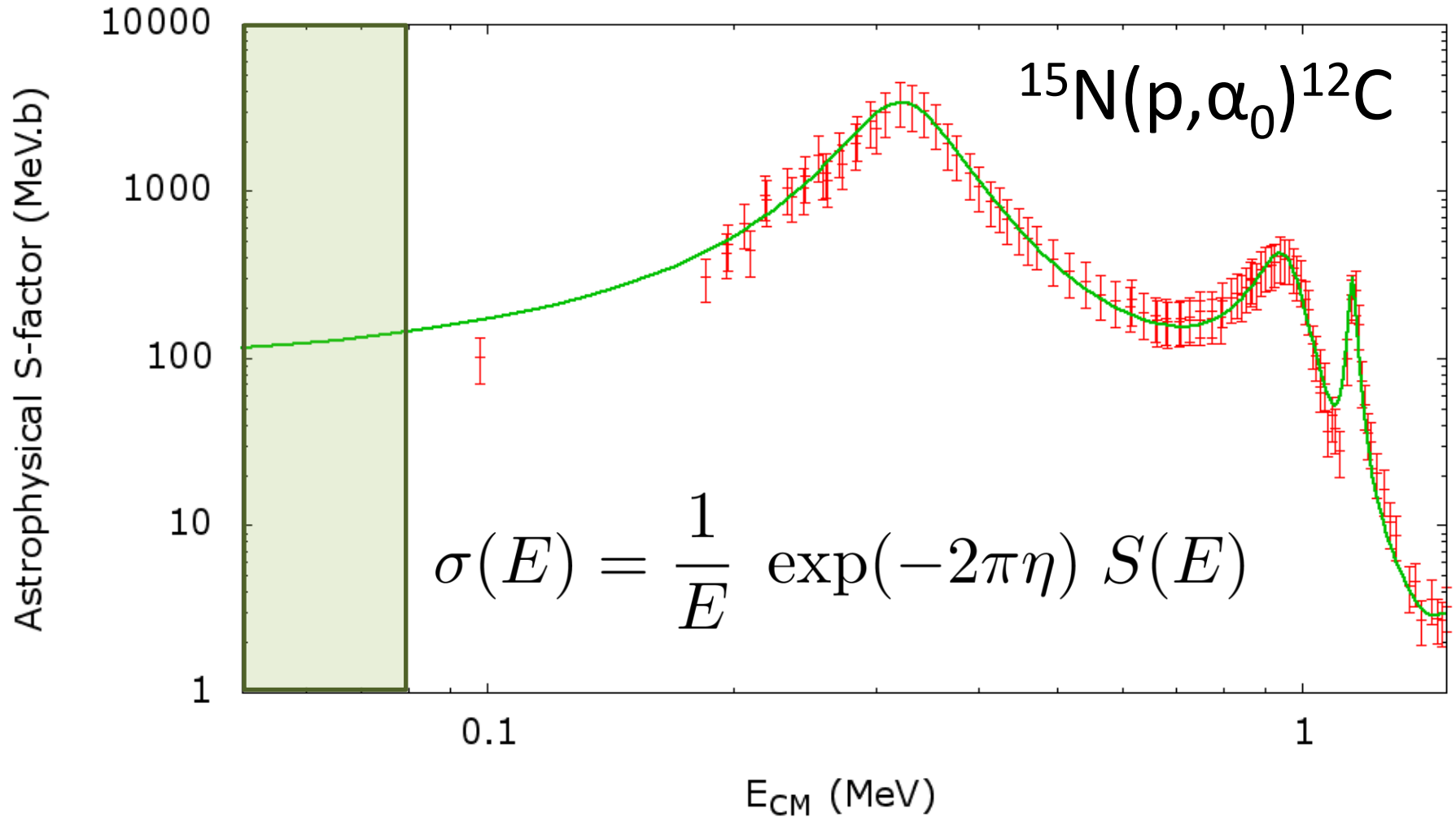


Nuclear astrophysics

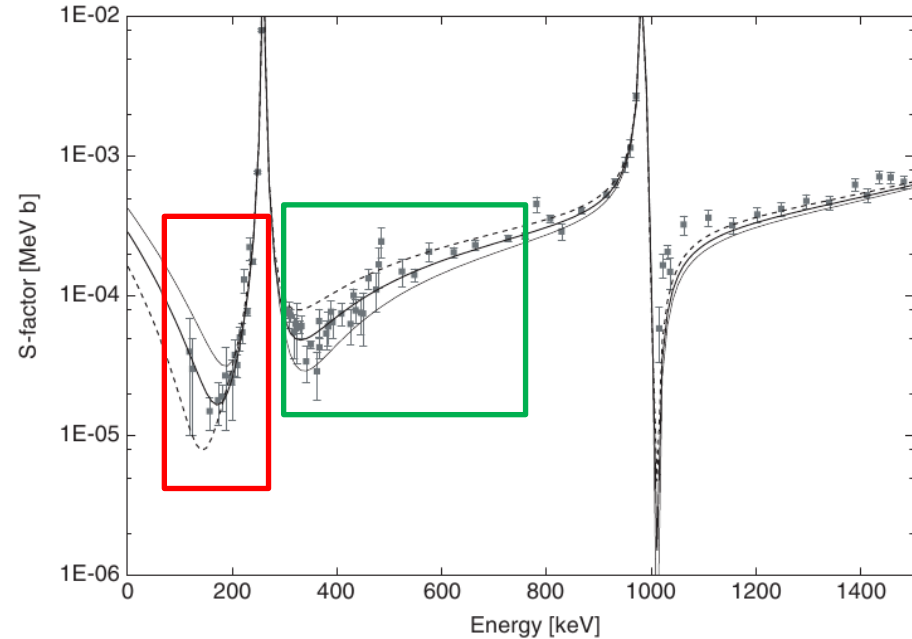
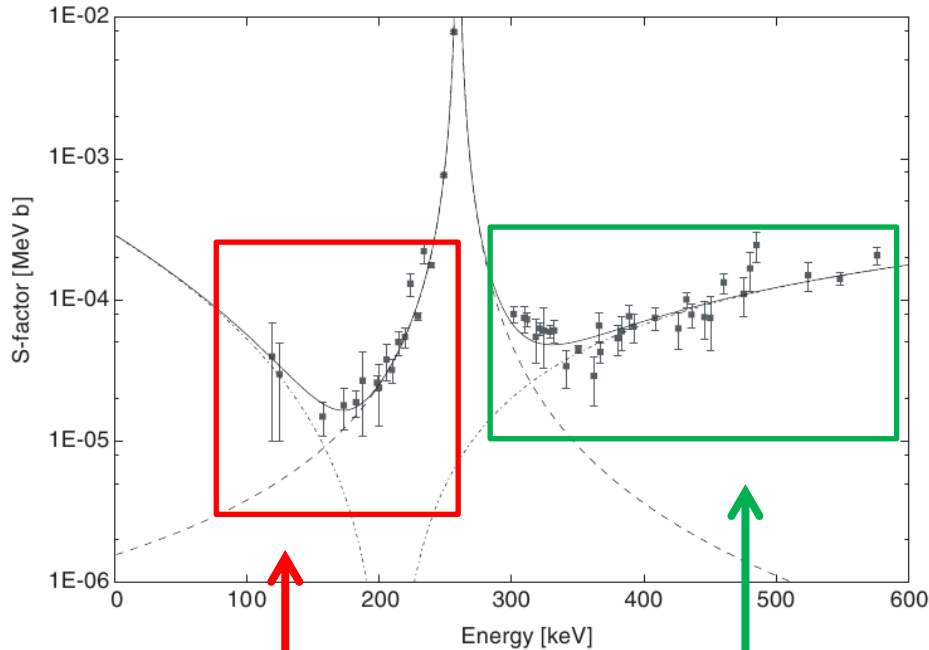
Gamow Window



The astrophysical S-factor



Guiding experiment



LUNA
(under a mountain)
Circa 2005

Münster/Toronto
Circa 1987

[Runkle et al., Phys. Rev Letts. 94, 082503 \(2005\)](#)

[Trautvetter et al., J. Phys. G 35, 014019 \(2008\)](#)

[Marta et al., Phys. Rev. C 78, 022802\(R\) \(2008\)](#)

[Adelberger et al., Rev. Mod. Phys. 83, 195 \(2012\)](#)

Low energy cross sections

- **We (usually) cannot measure the cross section...**
- **What determines the low-energy cross section?**
 - Tails of higher lying states?
 - Sub-threshold resonances?
 - Non-compound nuclear processes?
 - (Unobserved) resonances in the Gamow window?
- **What would help constrain this most?**
 - Direct measurements at higher energies?
 - Resonance widths? (From a different reaction?)
 - ANC measurements from transfer (for capture)?
 - Lifetime (and ANC) measurements for sub-threshold states?
 - Structure information from mirror nuclei?
 - New direct measurements of related channels?
 - New low energy direct measurements.....?

Materials analysis with ion beams

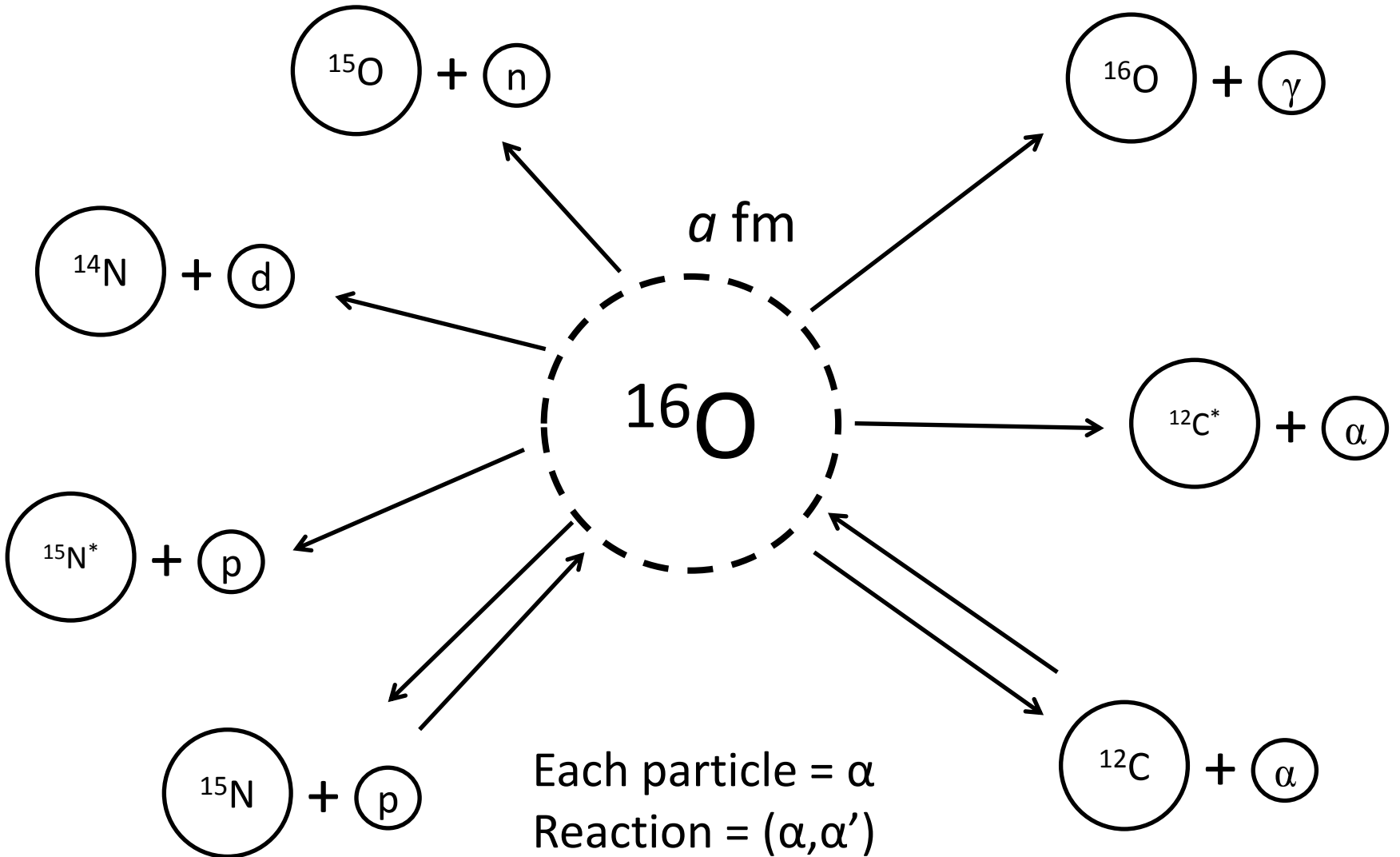
- **Scattering and reactions...**
 - Backward angle scattering cross sections
 - Reaction and capture cross sections
 - Wide variety of light-ion reactions on light-nuclei
 - Need to know at any energy and angle
- **What do we need?**
 - Scattering and reaction $\sigma(E,\theta)$...
 - Which angles and energies are most useful?
 - Which reactions are most important?
 - Understanding historic data sets? Verification experiments?
 - What nuclear data (energies and spin-parities) is available? Is this sufficient?
 - New scattering and reactions measurements?

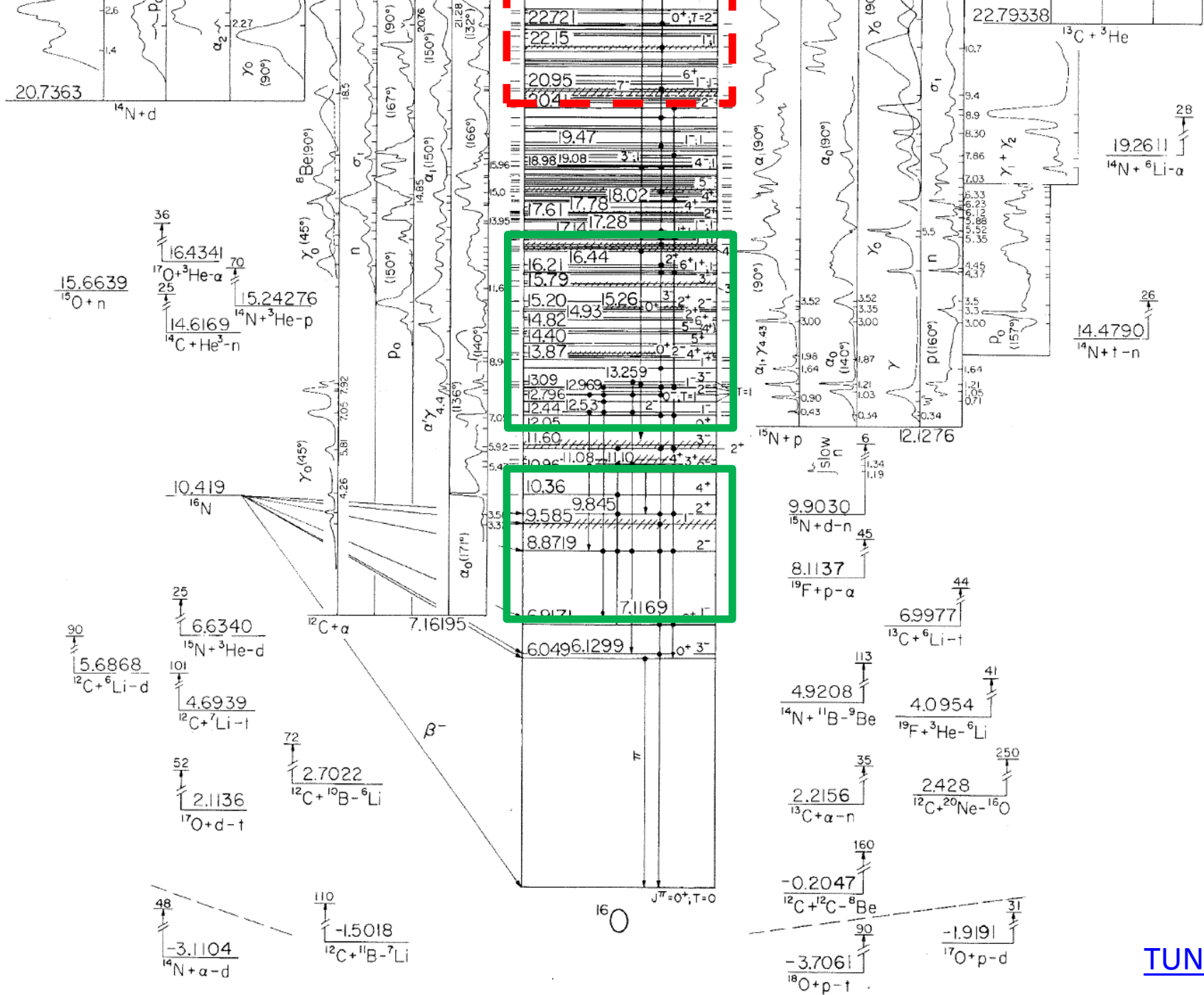
Resonance reactions and R-matrix phenomenology

General approach, definitions and resonance parameters



Example: ^{16}O compound reactions





Definitions

- α = Particle pair
- s = channel spin
- ℓ = relative angular momentum
- $c = \alpha s \ell$ = channel, a particular particle pair with channel spin s and angular momentum ℓ
- λ = compound nucleus state
- $\gamma_{\lambda c}$ = reduced width amplitude
- a = R-matrix radius, marks the division between internal and external regions

Channel spin and angular momentum

I_1 = Angular momentum of particle 1

I_2 = Angular momentum of particle 2

$$\vec{s} = \vec{I}_1 + \vec{I}_2$$

$$s = |I_1 - I_2| \dots I_1 + I_2$$

$$\vec{J} = \vec{s} + \vec{\ell}$$

$$J = |s - \ell| \dots s + \ell$$

where $\pi = \pi_1 \pi_2 (-1)^\ell$ is parity of resonance

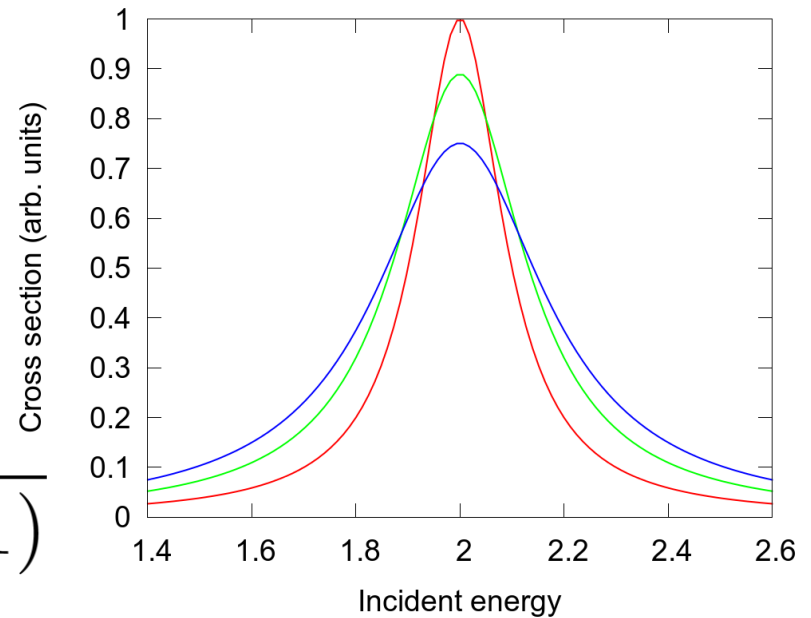
Breit-Wigner resonance

$$\sigma_{cc'}^{BW}(E) = \frac{\pi g_J}{k_c^2} \frac{\Gamma_c^o \Gamma_{c'}^o}{(E_R - E)^2 + (\Gamma^o/2)^2}$$

$$\Gamma^o = \sum_k \Gamma_k^o$$

$$\tau = \hbar/\Gamma^o$$

$$g_J = \frac{2J + 1}{(2I_1 + 1)(2I_2 + 1)}$$



- Lifetime = $\hbar/\Gamma^o = 10^{-16} - 10^{-21}$ seconds ($\Gamma^o = eV - MeV$)
- Direct reactions timescales $\sim 10^{-22}$ seconds

Breit-Wigner limits

- Half maximum at $(E_R \pm \Gamma^o/2)$
- If one width narrow e.g.

$\Gamma_c = \Gamma_p =$ particle width

$\Gamma_{c'} = \Gamma_g =$ gamma-ray width

$$\text{If } \Gamma_c^o \gg \Gamma_{c'}^o \quad \sigma_{cc'}^{BW}(E_R) \propto \frac{\Gamma_c^o \Gamma_{c'}^o}{(\Gamma_c^o + \Gamma_{c'}^o)^2} \approx \frac{\Gamma_{c'}^o}{\Gamma_c^o}$$

- In fitting, widths play different roles
 - Γ_c determines shape and width
 - $\Gamma_{c'}$ determines peak cross section
- Other channels (i.e. $\Gamma_{\text{total}} = \Gamma_c + \Gamma_{c'} + \Gamma_x$) broaden the resonance and reduce the peak cross section

R-matrix parameters

- Level energies and reduced width amplitudes

$$R_{cc'}^J = \sum_{\lambda} \frac{\gamma_{\lambda c' J} \gamma_{\lambda c J}}{E_{\lambda} - E}$$

Treat **energies** and **widths** as parameters and fit to reproduce data

$$\gamma_{\lambda c J} = \left(\frac{\hbar^2}{2\mu_c a_c} \right)^{1/2} \int dS \varphi_c^* X_{\lambda J M}$$

$$H X_{\lambda J M} = E_{\lambda J} X_{\lambda J M}$$

- Radius a_c [and (arbitrary) boundary condition B_c]
- How can we relate to Breit-Wigner-type parameters?

R-matrix cross section

- Angle-integrated cross section:

$$\sigma_{\alpha\alpha'} = \frac{\pi g_J}{k_\alpha^2} \sum_{J\ell\ell' s s'} \left| e^{2i\omega_c} \delta_{cc'} - U_{cc'} \right|^2$$

- Matrix expression for collision (scattering) matrix:

$$U = \Omega \left[\mathbf{1} + \mathbf{P}^{\frac{1}{2}} \frac{\mathbf{R}}{\mathbf{1} - \mathbf{R}\mathbf{L}^0} \mathbf{P}^{\frac{1}{2}} \mathbf{w} \right] \Omega$$

$$\mathbf{L}^0 = \mathbf{S} + i\mathbf{P} - \mathbf{B}$$

Single resonance XS (via A-matrix)

$$\sigma_{cc'}(E) = \frac{\pi g_J}{k_c^2} \frac{\Gamma_{\lambda c} \Gamma_{\lambda c'}}{(E_\lambda + \Delta_\lambda - E)^2 + (\Gamma_\lambda/2)^2}$$

$$\Gamma_{\lambda c} = 2P_c \gamma_{\lambda c}^2$$

$$\Delta_\lambda = - \sum_k \gamma_{\lambda k}^2 \left[S_k(E) - B_k \right]$$

Arbitrary boundary condition

$$\Gamma_\lambda = \sum_k \Gamma_{\lambda k}$$

Shift-function

$$g_J = \frac{2J + 1}{(2I_1 + 1)(2I_2 + 1)}$$

Lane and Thomas, page 327, [Rev. Mod. Phys. 30, 257 \(1958\)](#)

Carl Brune <http://arxiv.org/abs/nucl-th/0502087>

Decouvemont and Baye, [Rep. Prog. Phys 73, 036301 \(2010\)](#)

Observed and formal parameters

- Thomas approximation: choose $B_k = S(E_\lambda)$, then Taylor expand $S(E)$ to define one relationship between physical (BW) and formal (R-matrix) parameters

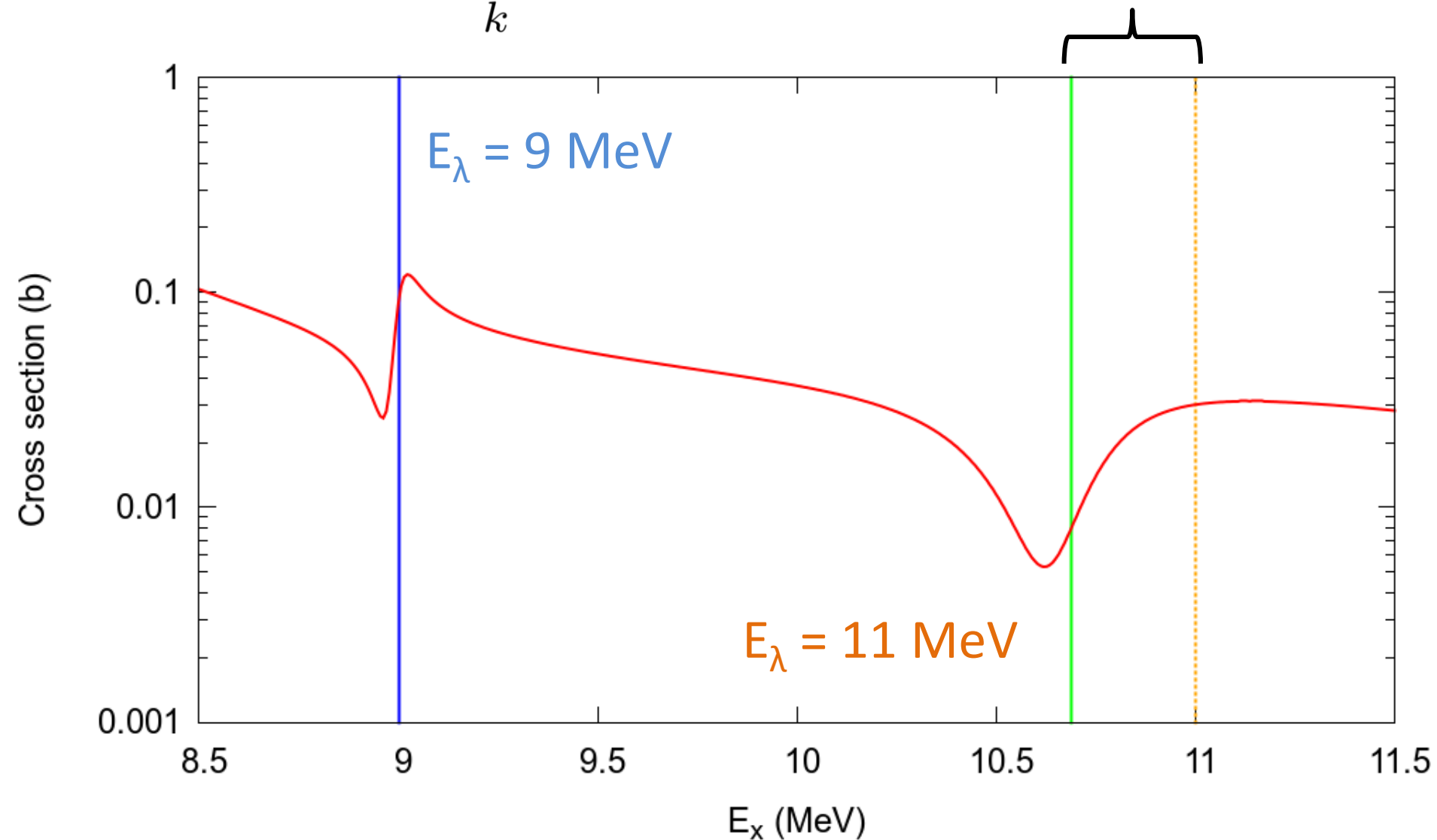
$$\Gamma_{\lambda c}^o = \frac{2P_c \tilde{\gamma}_{\lambda c}^2}{1 + \sum_k \tilde{\gamma}_{\lambda k}^2 \left(\frac{dS_k}{dE} \right) \tilde{E}_\lambda} \quad \text{Formal}$$

Observed

$$\tilde{\gamma}_{\lambda c}^2 = \frac{\Gamma_{\lambda c}^o}{P_c} \left[2 - \sum_k \frac{\Gamma_{\lambda k}^o}{P_k} \left(\frac{dS_k}{dE} \right) \tilde{E}_\lambda \right]^{-1}$$

“On-resonance” parameters

$$\Delta_\lambda = - \sum_k \tilde{\gamma}_{\lambda k}^2 [S_k(E) - B_k] \approx -0.31 \text{ MeV}$$



Alternative parameterization

- Alternative parameterization

$$\{\tilde{E}_\lambda, \tilde{\gamma}_{\lambda c}\} \rightarrow \{E_\lambda, \gamma_{\lambda c}, B_c\}$$

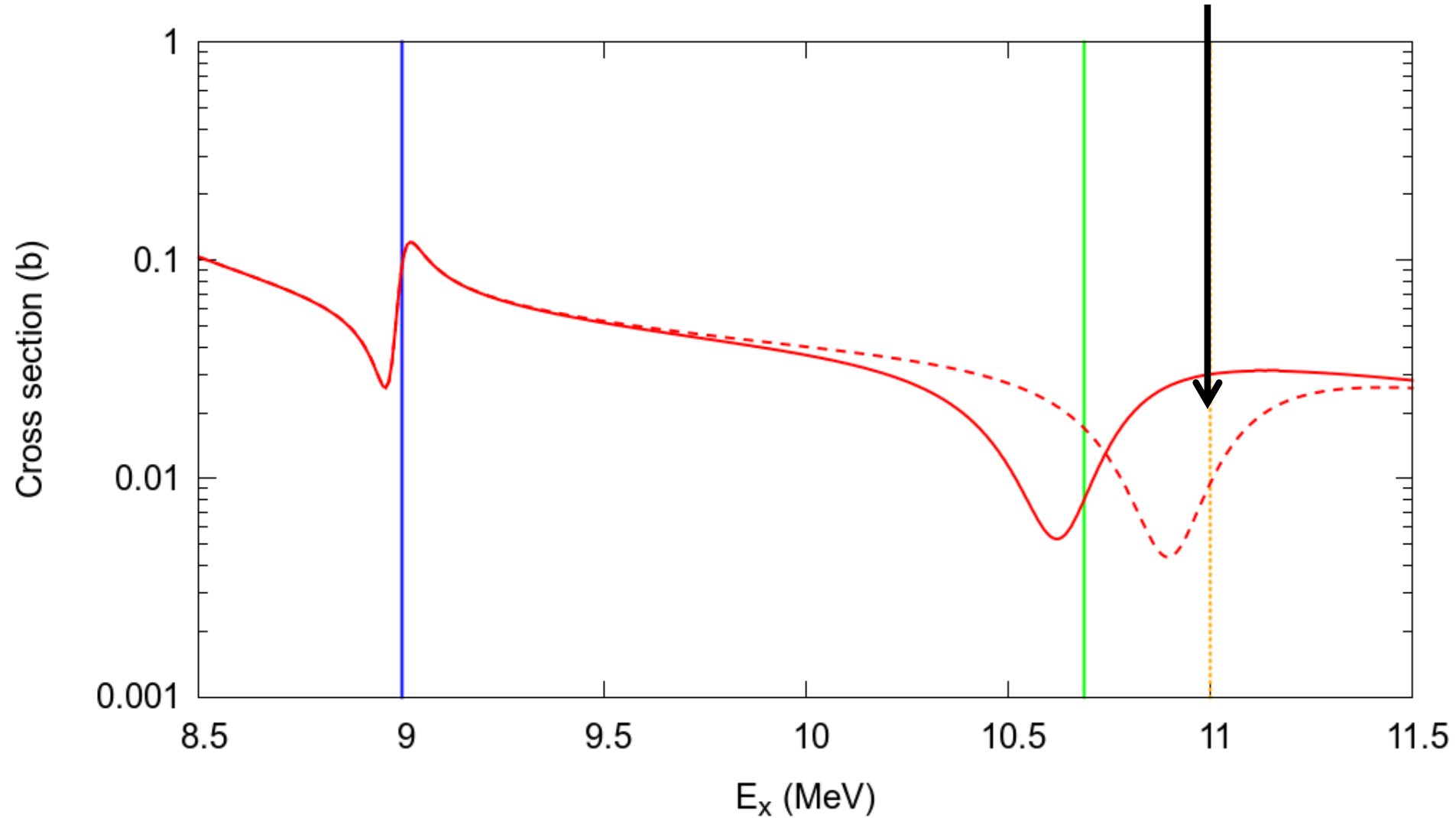
- Inputting \uparrow far easier
- Also write cross section in terms of alternative parameters – no boundary condition
- In fitting, can now fix some energies and widths

Angulo and Descouvemont, [Phys. Rev. C 61, 064611 \(2000\)](#)

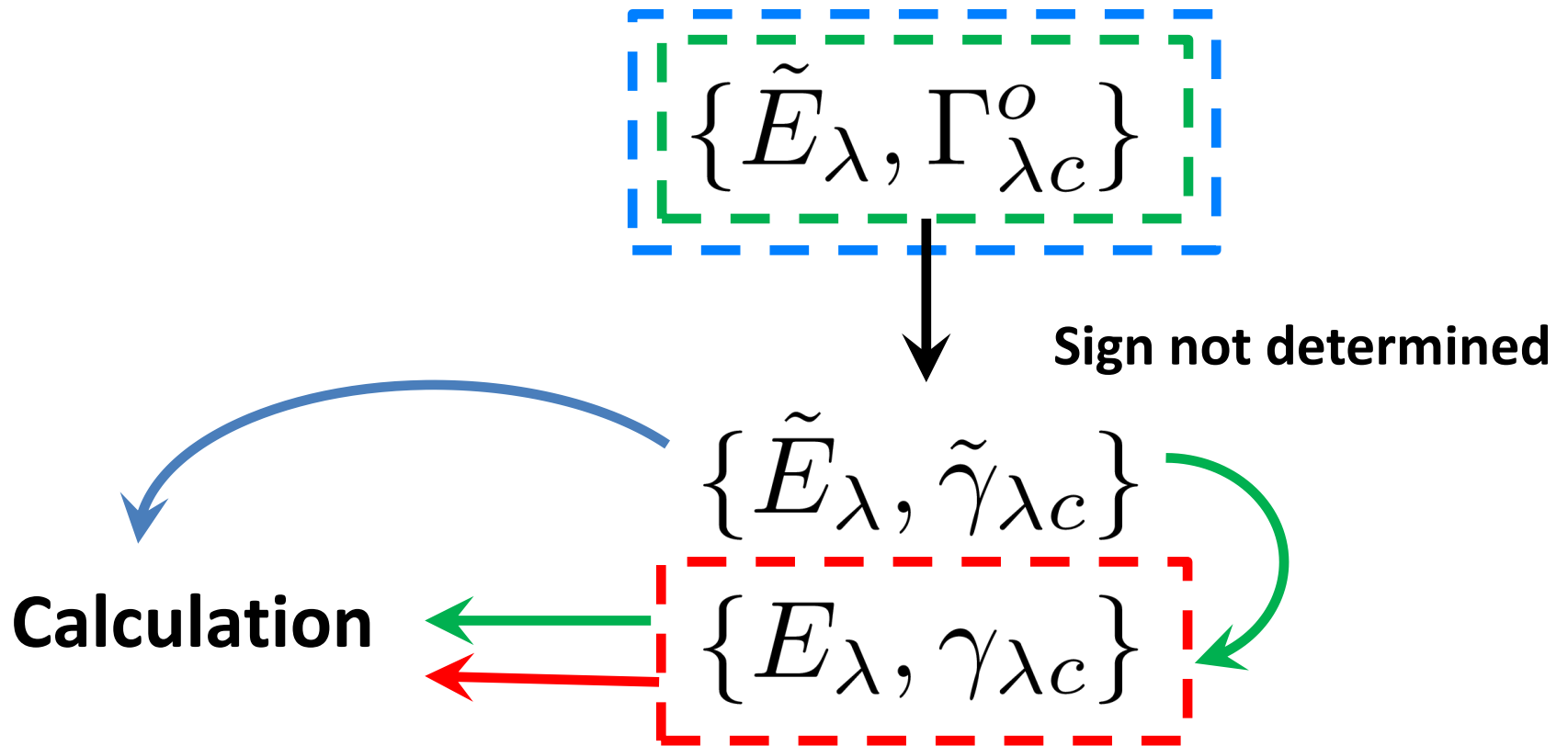
Carl Brune, [Phys. Rev. C 66, 044611 \(2002\)](#)

Descouvemont and Baye, page 28/29, [Rep. Prog. Phys 73, 036301 \(2010\)](#)

Alternative parameterization allows you to input on-resonance energies and widths for ALL states



Parameter Input in AZURE



Where a boundary condition required it is always set to the shift function at energy of the lowest level for each spin-parity

Fixing parameters

Only if all parameters of a given resonance are fixed will they all not change in fitting

$$\Gamma_{\lambda c}^o = \frac{2P_c \tilde{\gamma}_{\lambda c}^2}{1 + \sum_k \tilde{\gamma}_{\lambda k}^2 \left(\frac{dS_k}{dE} \right) \tilde{E}_\lambda}$$

$$\tilde{\gamma}_{\lambda c}^2 = \frac{\Gamma_{\lambda c}^o}{P_c} \left[2 - \sum_k \frac{\Gamma_{\lambda k}^o}{P_k} \left(\frac{dS_k}{dE} \right) \tilde{E}_\lambda \right]^{-1}$$

But you should be able to more easily establish a good set of starting parameters

The Wigner limit

- Defines maximum value of formal reduced width amplitude

$$\gamma_{Wc}^2 = \frac{3\hbar^2}{2\mu_c a_c^2} \quad \sim \text{MeV}$$

$$\theta_{\lambda c}^2 = \gamma_{\lambda c}^2 / \gamma_{Wc}^2$$

- Values of θ near 1 (i.e. γ large fraction of γ_W) indicates cluster-type states, normally much smaller
- Dependent on the R-matrix radius a
- Γ_W varies with energy due to penetrability

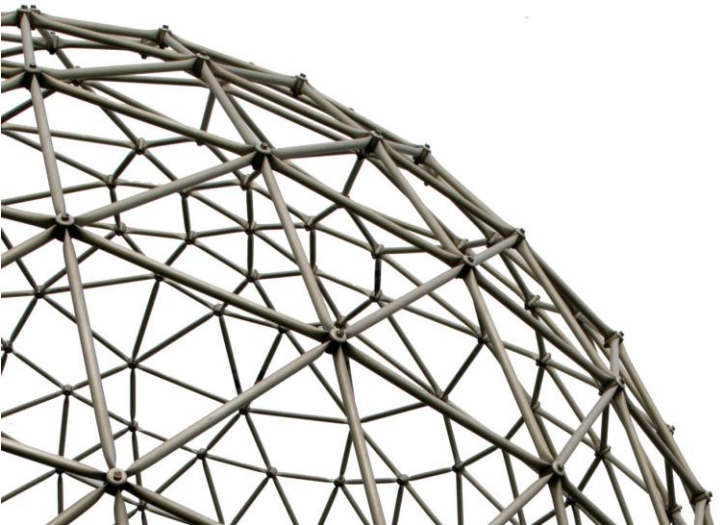
Teichmann and Wigner, [Phys. Rev. 87, 123 \(1952\)](#)

Decouvemont and Baye, [Rep. Prog. Phys 73, 036301 \(2010\)](#)

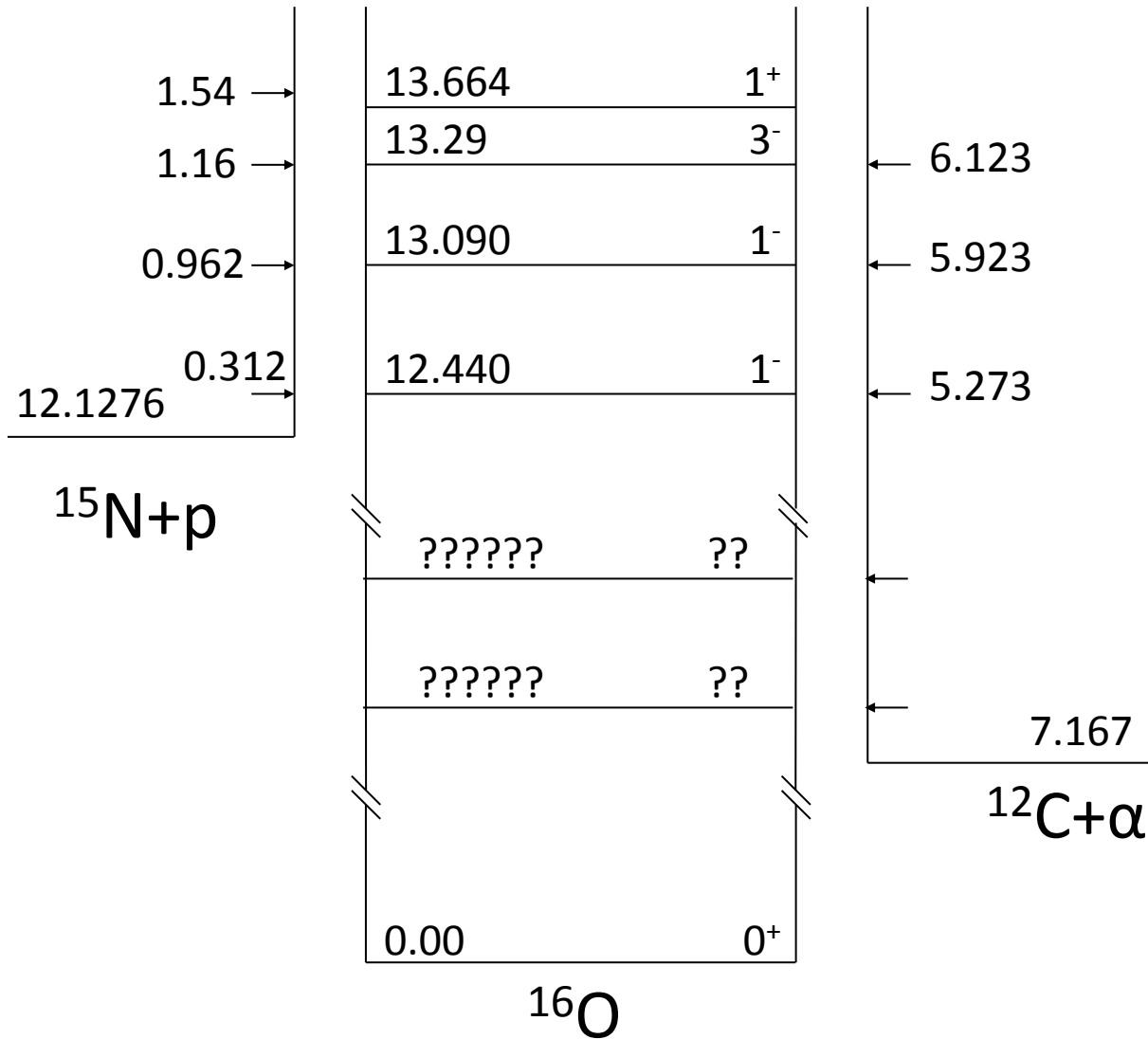
Freer *et al.*, [Phys. Rev. C 85, 014304 \(2012\)](#)

Analysis of elastic scattering and reactions

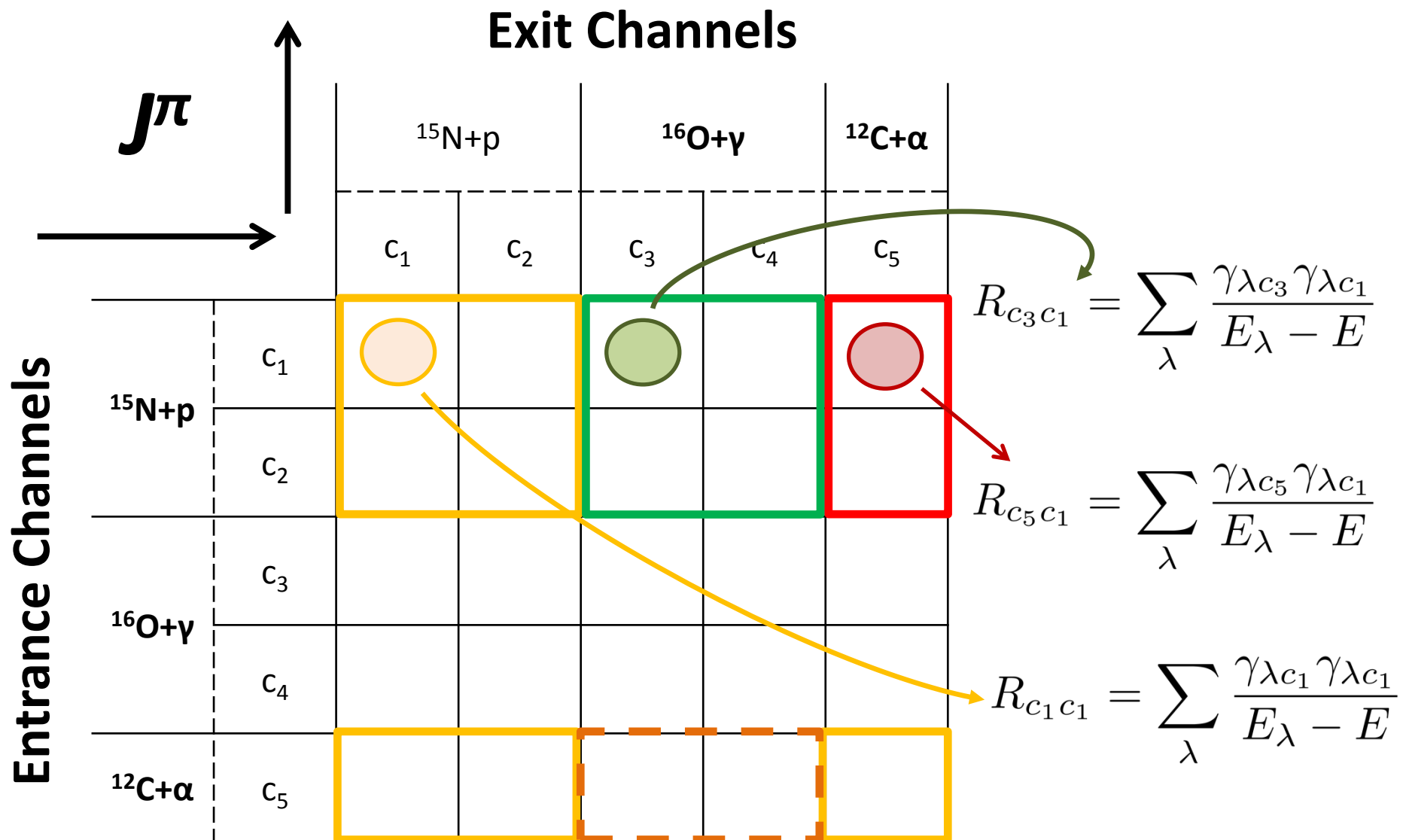
Alpha-particle scattering on ^{12}C
Closing the CNO cycle: $^{15}\text{N}(p,\alpha)^{12}\text{C}$



^{16}O thresholds: $^{12}\text{C}(\alpha,\alpha)^{12}\text{C}$, $^{15}\text{N}(p,\alpha)^{12}\text{C}$



^{16}O R-matrix: $^{15}\text{N}(p,\gamma)^{16}\text{O}$



Channels for $^{12}\text{C}+\alpha$

- Calculated automatically by AZURE2
- For $^{12}\text{C}+\alpha$ things are simple:

$$I_1 = 0 \quad \pi_1 = 1$$

$$I_2 = 0 \quad \pi_2 = 1$$

$$\therefore s = 0$$

$$\text{and } J = \ell \quad \pi = \pi_1 \pi_2 (-1)^\ell = (-1)^\ell$$

- One channel per J , only natural parity states allowed $\pi = (-1)^J$
- In general, many channels allowed per resonance

Hard-sphere phase shift

- For elastic scattering, must include all hard sphere contributions

$$U = \Omega \left[1 + P^{\frac{1}{2}} \frac{R}{1 - RL^0} P^{\frac{1}{2}} w \right] \Omega$$



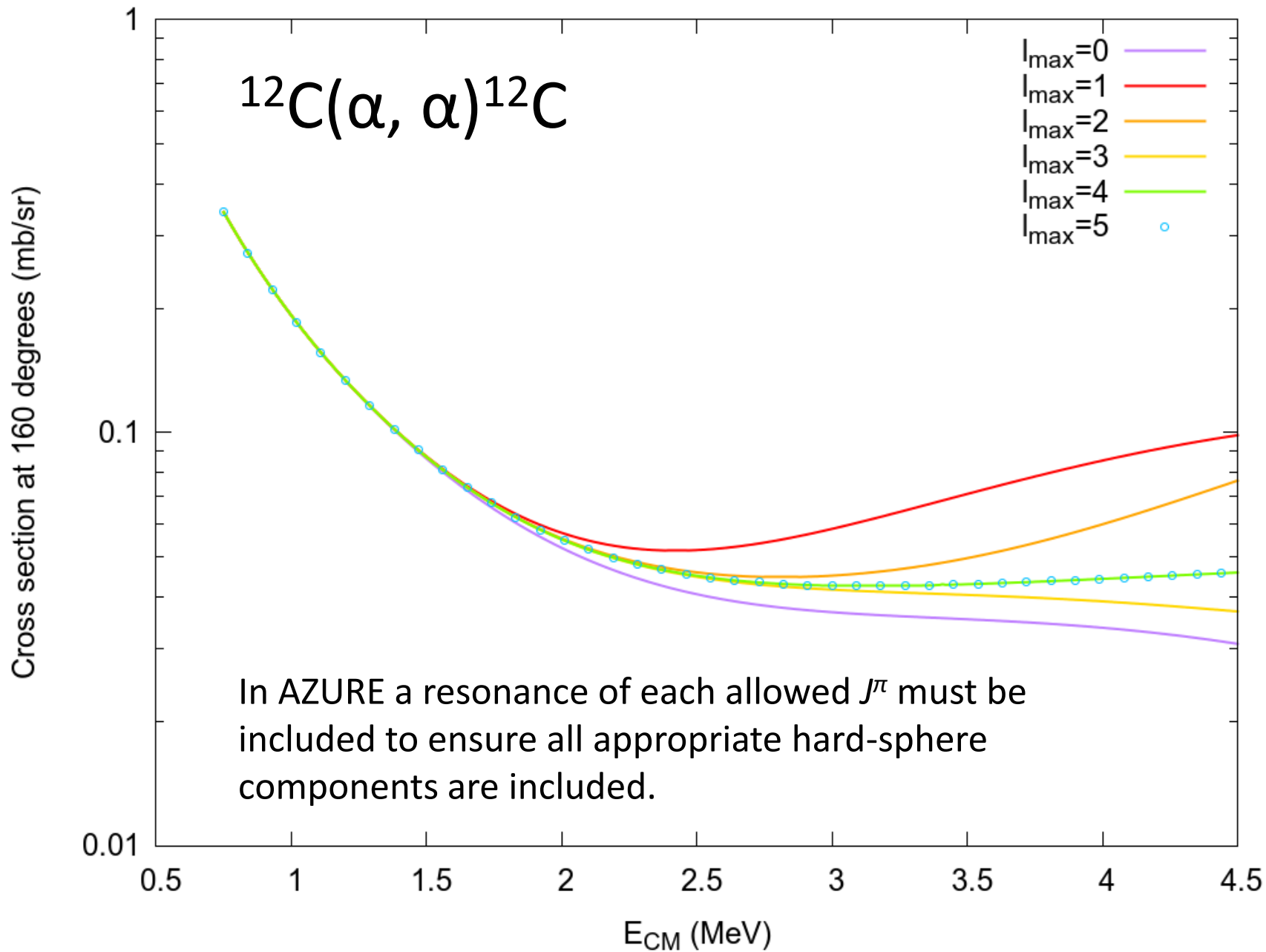
$$\Omega_c^+ = \exp[i(\omega_c - \phi_c^+)]$$

$$\delta_{HS} = -\phi_c^+ = -\tan^{-1}(F_c/G_c)$$

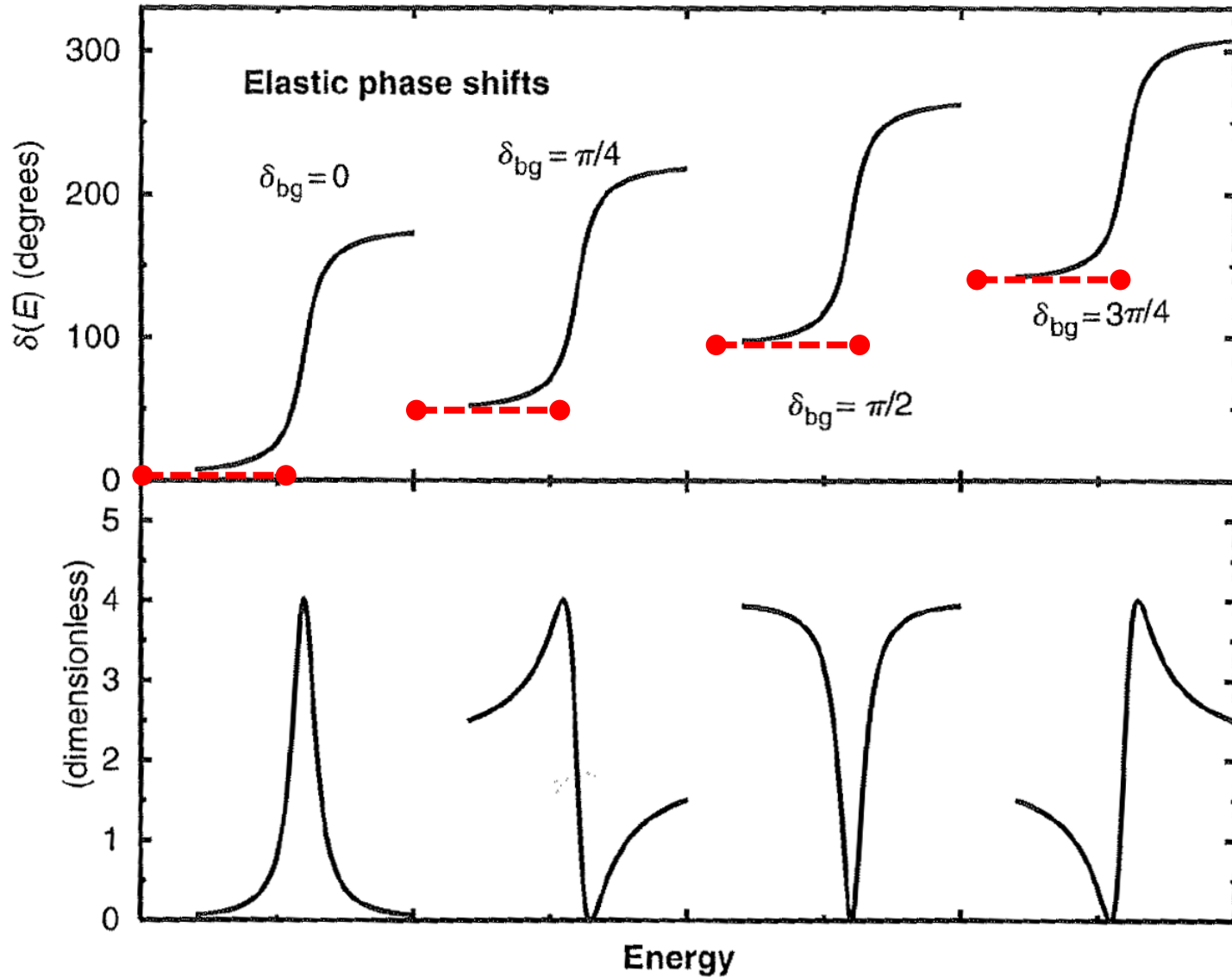
L. V. Namjoshi *et al.*, PRC 13, 915 (1976)

Ruiz *et al.*, PRC 71, 025802 (2005)

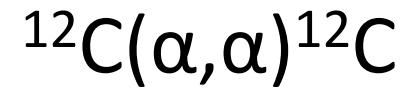
Lane and Thomas, page 271 and 289, [Rev. Mod. Phys. 30, 257 \(1958\)](#)



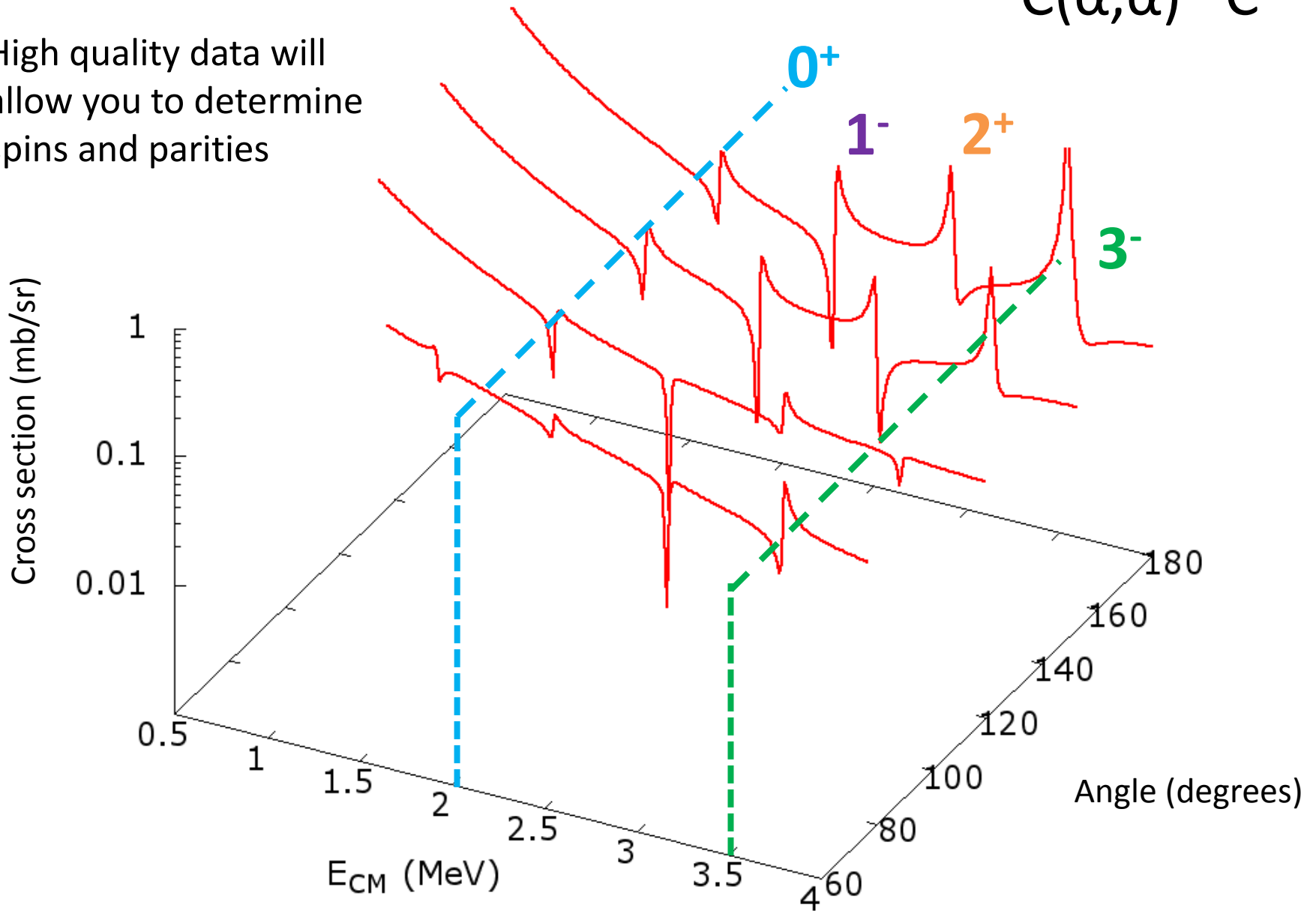
Resonance shapes



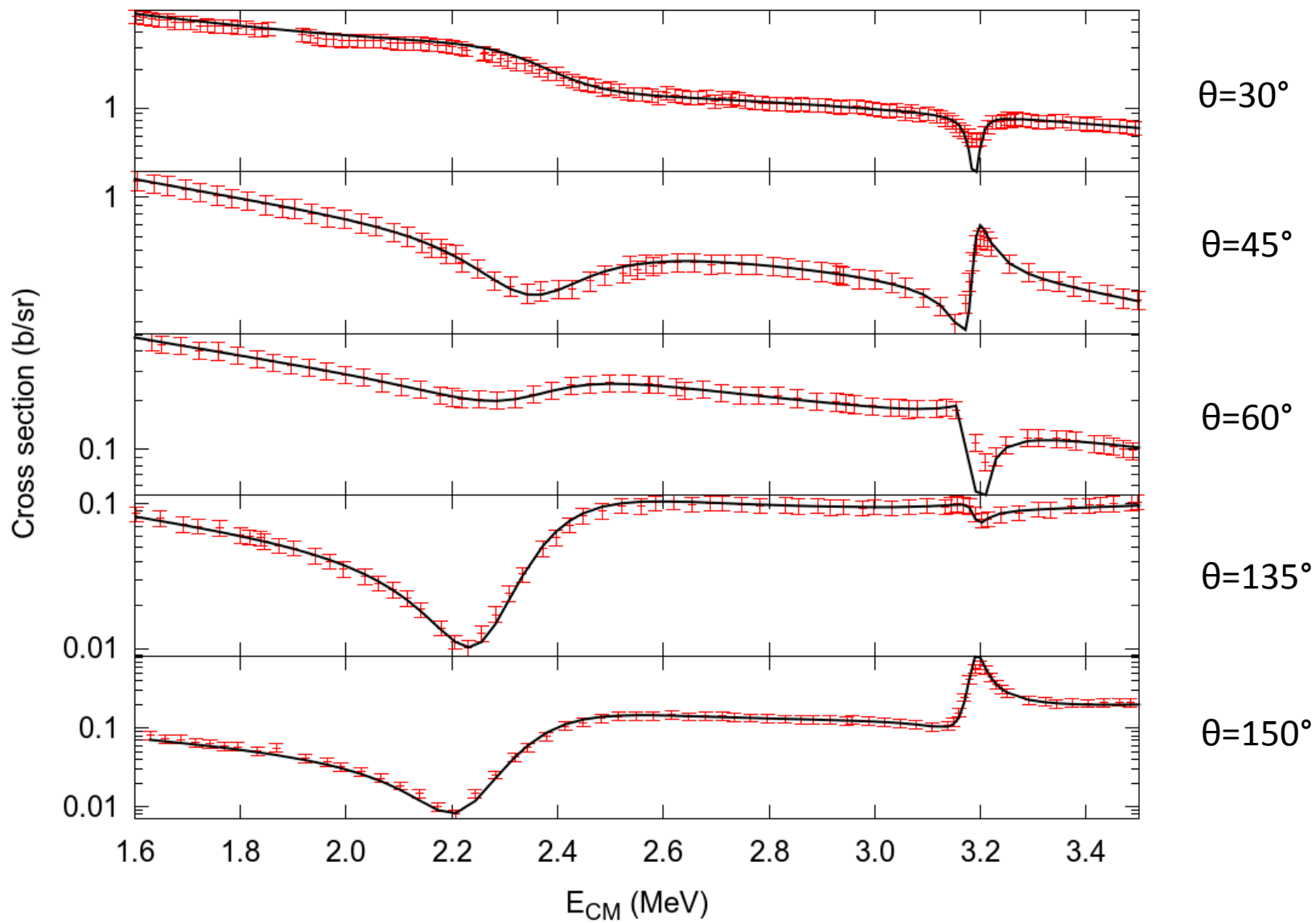
Angular distributions



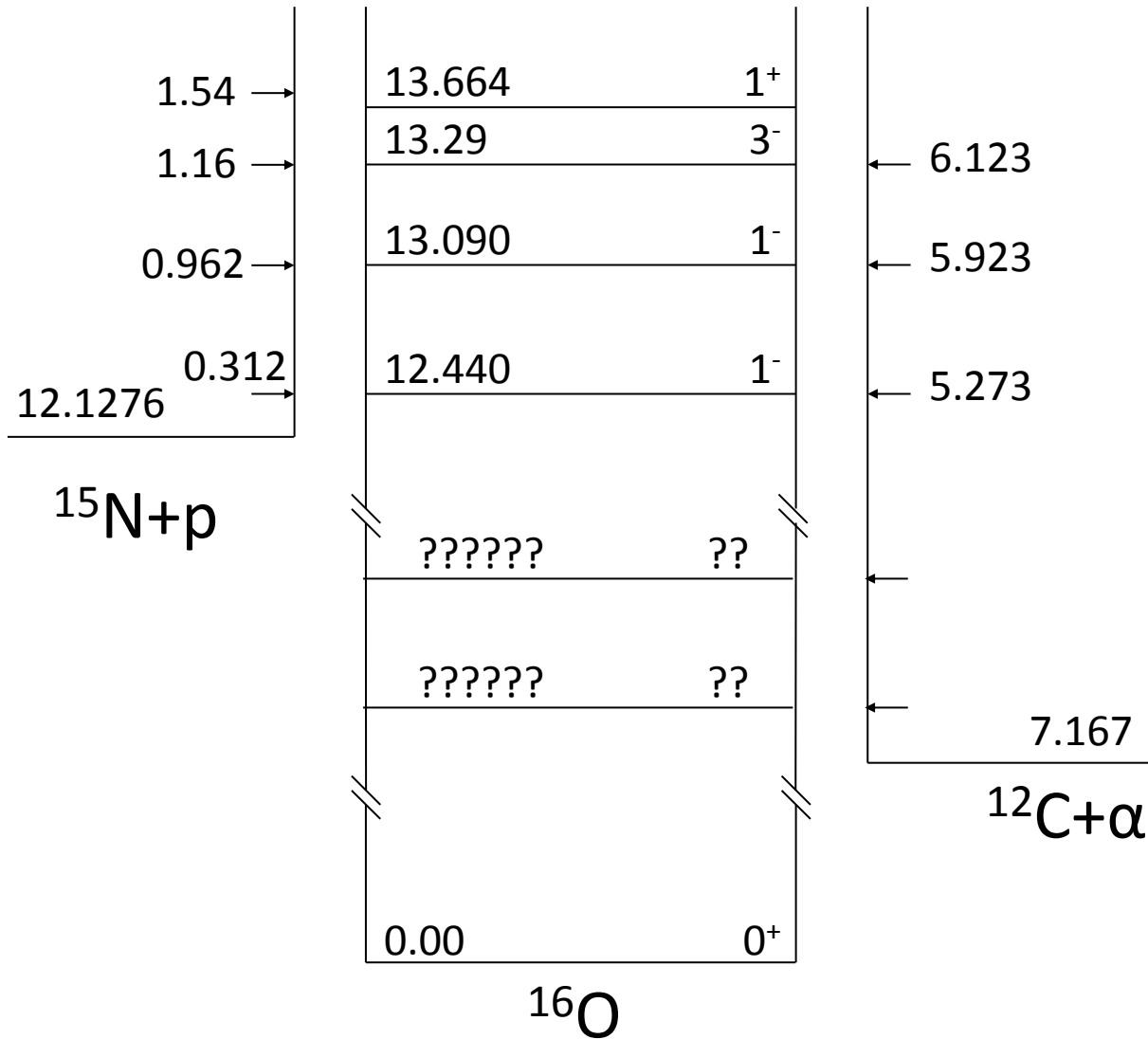
High quality data will allow you to determine spins and parities



$^{12}\text{C}(\alpha,\alpha)^{12}\text{C}$



^{16}O thresholds: $^{12}\text{C}(\alpha,\alpha)^{12}\text{C}$, $^{15}\text{N}(p,\alpha)^{12}\text{C}$



Proton channels in $^{15}\text{N}(p,\alpha)^{12}\text{C}$

$$I_1 = 1/2, \quad \pi_1 = +$$

$$I_2 = 1/2, \quad \pi_2 = -$$

$$\therefore s = 0, 1$$

e.g. $J^\pi = 0^+$, $^{15}\text{N}+p$ $s = 1, \ell = 1$

$$J^\pi = 1^-, \quad s = 1, \ell = 0$$

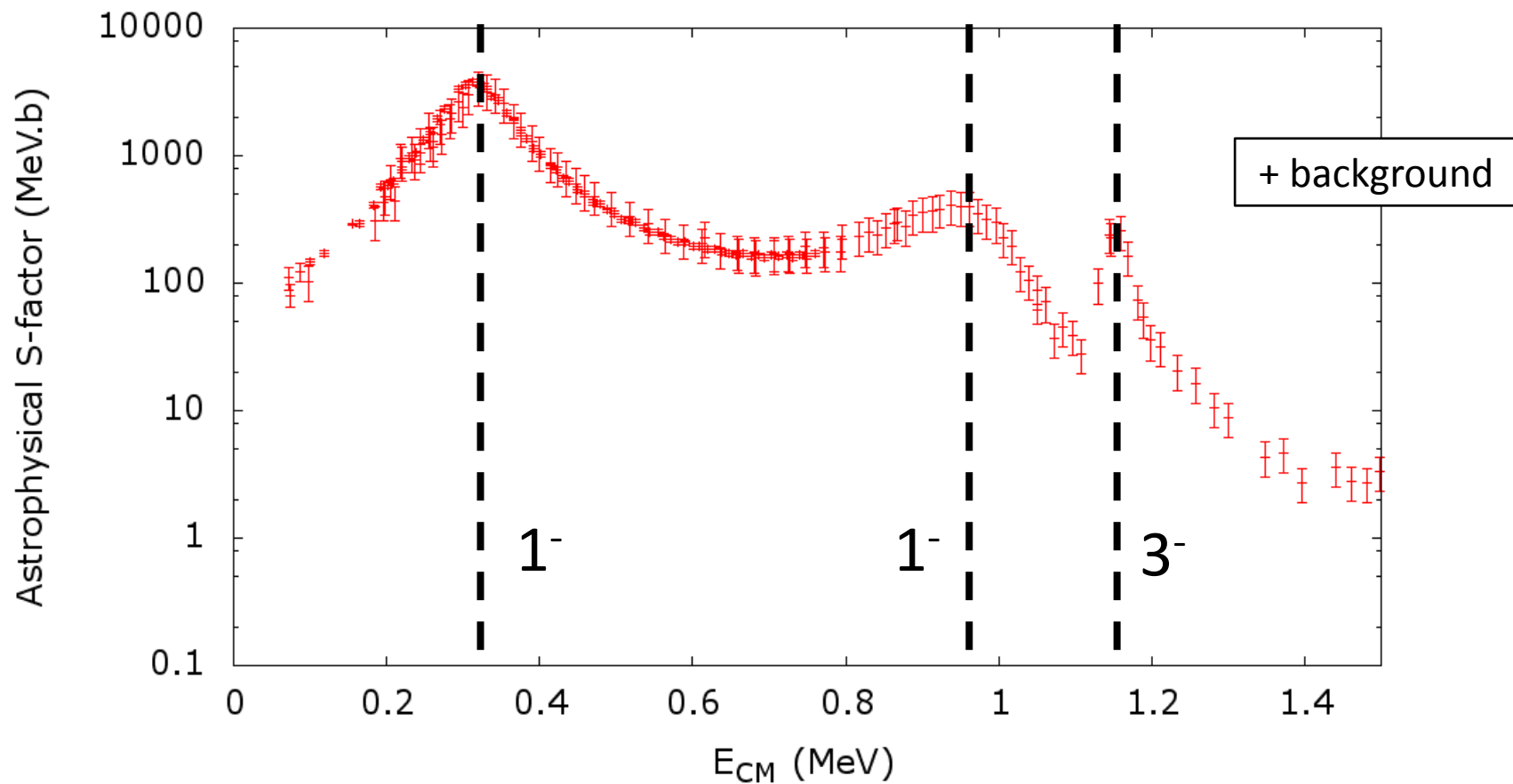
$$s = 1, \ell = 2$$

$$J^\pi = 2^+, \quad s = 1, \ell = 1$$

$$s = 1, \ell = 3$$

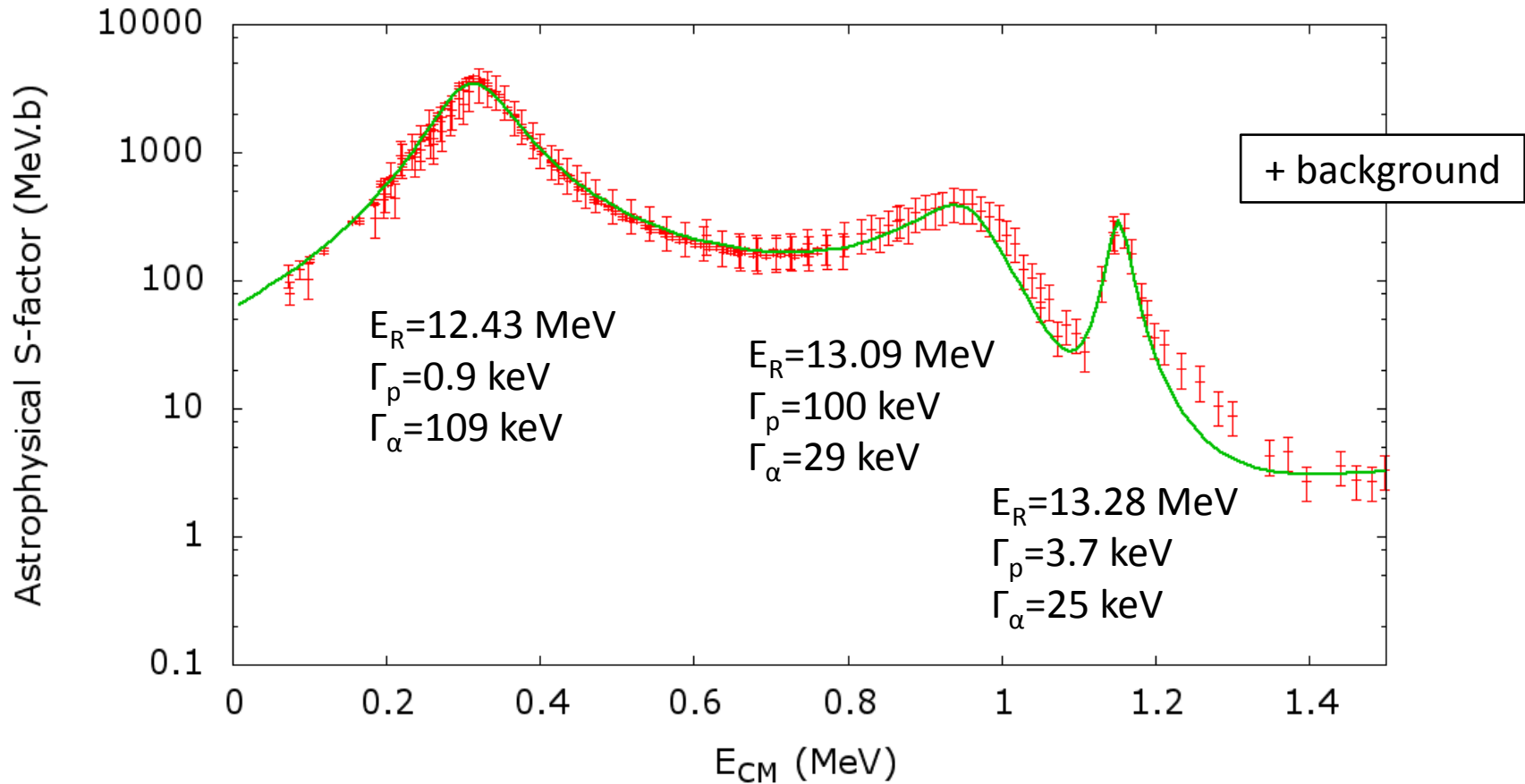
- $s=0$ doesn't occur for natural parity resonances

$^{15}\text{N}(p,\alpha)^{12}\text{C}$

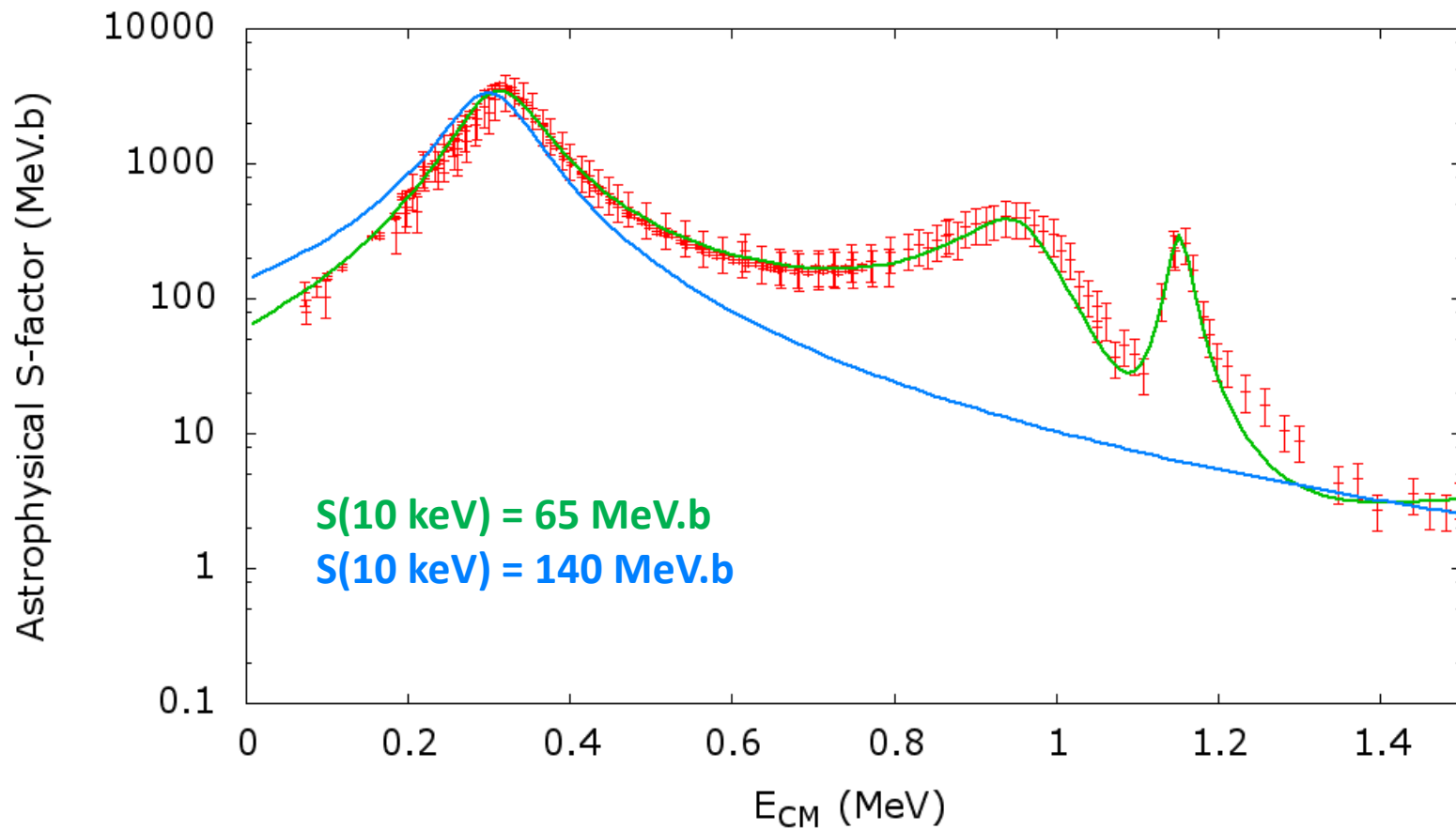


$^{15}\text{N}(p,\alpha)^{12}\text{C}$

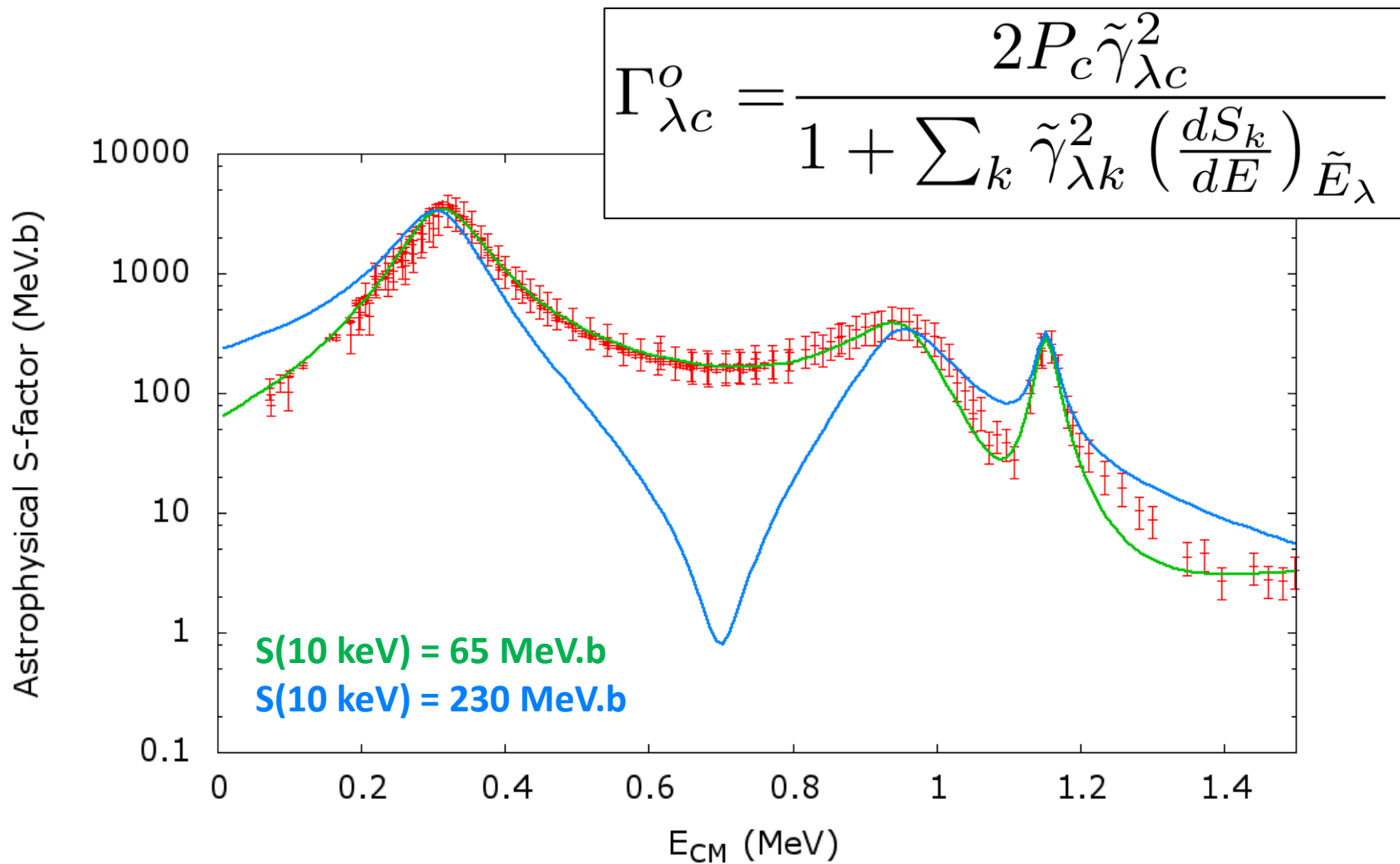
3 states, 9 parameters (assuming lowest ℓ)
+ background resonance parameters



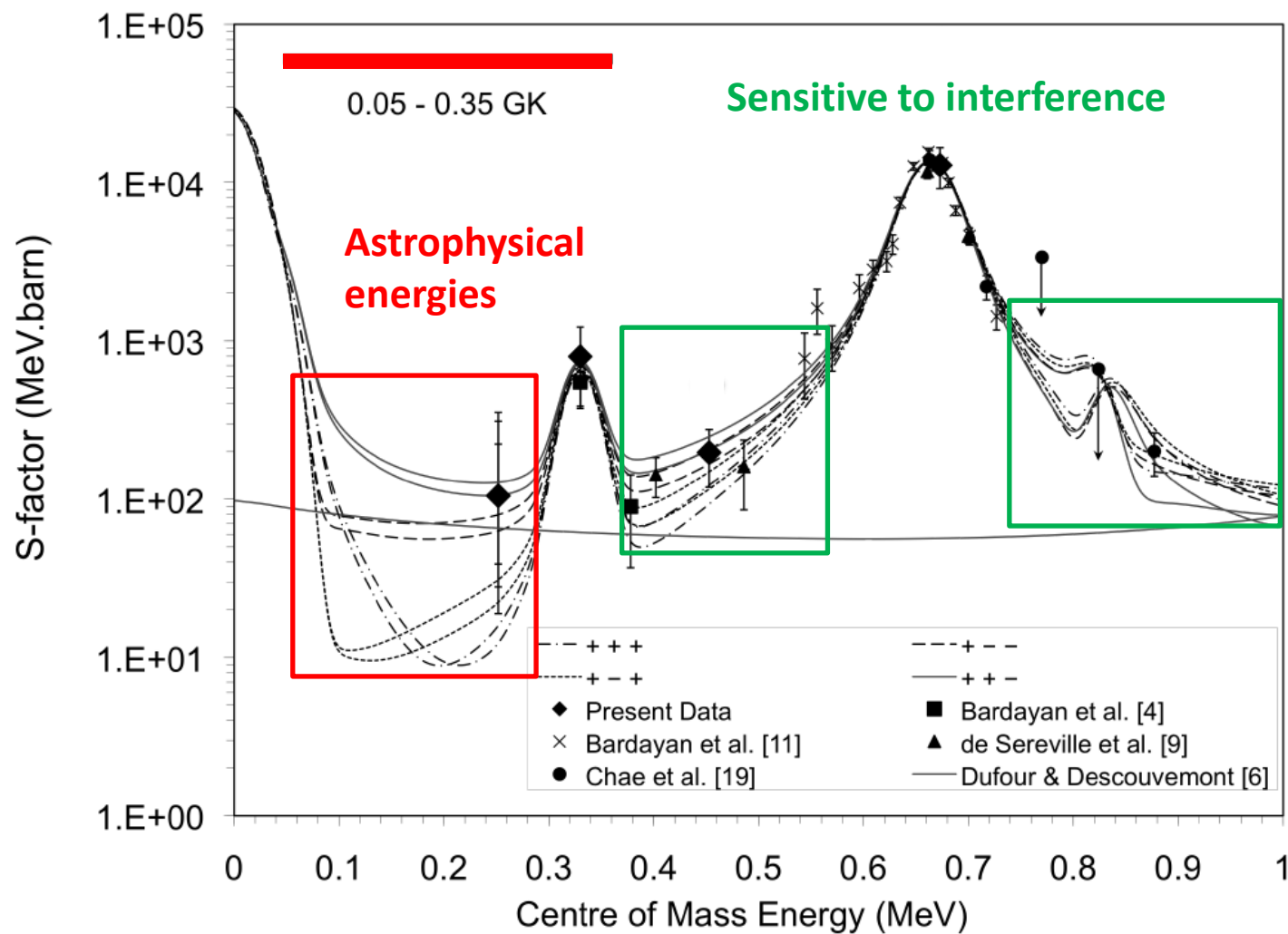
$^{15}\text{N}(p,\alpha)^{12}\text{C}$



$^{15}\text{N}(p,\alpha)^{12}\text{C}$



$^{18}\text{F}(p,\alpha)^{15}\text{O}$ - interferences



Summary

R-matrix may be used to parameterize cross sections, allowing you to calculate the cross section where you do not have data (e.g. in astrophysics, materials analysis) and extract resonance properties.

Alternative parameterizations of the R-matrix exist to relate formal reduced width amplitudes arising from integrals over the compound nucleus surface to “observed”-type widths Γ .

R-matrix can be used to understand the essential structure and elements determining the cross section at astrophysical energies. This can help guide future experiments.