Scattering theory I: single channel differential forms

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equations of motion

laboratory

$$\left[-\frac{\hbar^2}{2m_A}\nabla^2_{\mathbf{r}_A} - \frac{\hbar^2}{2m_B}\nabla^2_{\mathbf{r}_B} + V(\mathbf{r}_A - \mathbf{r}_B) - E_{\text{tot}}\right]\Psi(\mathbf{r}_A, \mathbf{r}_B) = 0.$$

Center of mass

$$\left[-\frac{\hbar^2}{2m_{AB}}\nabla^2_{\mathbf{S}} - \frac{\hbar^2}{2\mu}\nabla^2_{\mathbf{R}} + V(\mathbf{R}) - E_{\text{tot}}\right]\Psi(\mathbf{S}, \mathbf{R}) = 0.$$

$$\Psi(\mathbf{S}, \mathbf{R}) = \Phi(\mathbf{S})\psi(\mathbf{R})$$

$$\Phi(\mathbf{S}) = A\exp(i\mathbf{K}\cdot\mathbf{S})$$

$$-\frac{\hbar^2}{2m_{AB}}\nabla^2_{\mathbf{S}}\Phi(\mathbf{S}) = (E_{\text{tot}} - E)\,\Phi(\mathbf{S})$$

and $\left[-\frac{\hbar^2}{2\mu}\nabla^2_{\mathbf{R}} + V(\mathbf{R})\right]\psi(\mathbf{R}) = E\,\psi(\mathbf{R})$

cross section

Definition of cross section:

the area within which a projectile and a target will interact and give rise to a specific product.

Units 1b (barn) = 10 fm x 10 fm

The number of particle entering a detector depends on:

- flux of the incident beam
- number of scattering centers in the target
- solid angular size of detector
- the cross sectional area for the reaction to occur

$$\frac{dN}{dt} = j_i \, n \, \Delta\Omega \, \sigma$$

$$\mathbf{j} = \mathbf{v}|\psi|^2$$

cross section

If we consider just one scattering center $n = 1$, and measure the scattered *angular flux* in the final state as $\hat{j}_f(\theta, \phi)$ particles/second/steradian, then

$$\sigma(\theta, \phi) = \frac{\hat{j}_f(\theta, \phi)}{j_i}$$

cross section in c.m. and lab

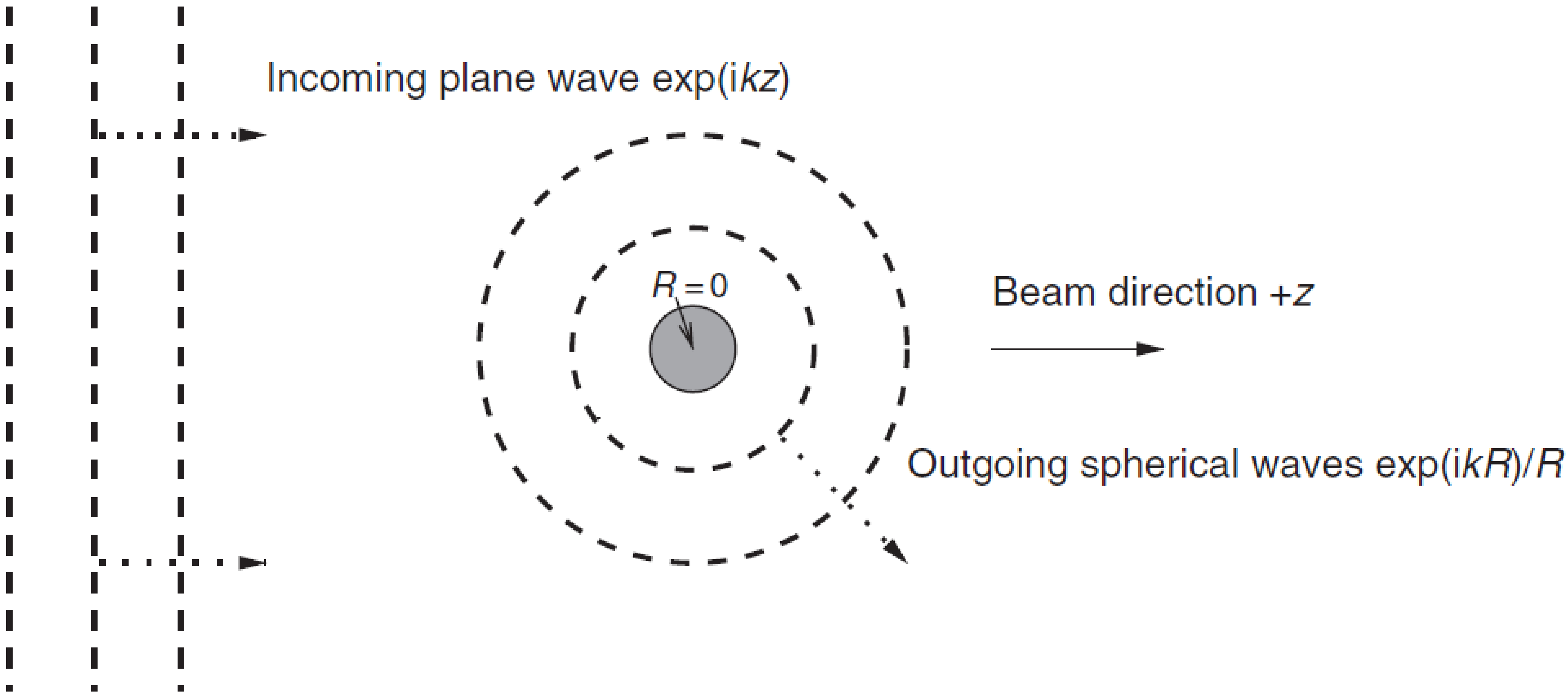
Total cross section:

the same in center of mass and laboratory

Angular distribution of the cross section:

$$\sigma(\theta,\phi)\,\mathrm{d}\phi\;\sin\theta\,\mathrm{d}\theta = \sigma_{\mathrm{lab}}(\theta_{\mathrm{lab}},\phi_{\mathrm{lab}})\,\mathrm{d}\phi_{\mathrm{lab}}\;\sin\theta_{\mathrm{lab}}\mathrm{d}\theta_{\mathrm{lab}}$$

picture for scattering



Scattering theory: setting up

Incoming beam

$$\psi^{\text{beam}} = A\exp(\mathrm{i}\mathbf{k}_i \cdot \mathbf{R})$$

$$\psi^{\text{beam}} = A\mathrm{e}^{\mathrm{i}k_i z}$$

Incoming flux

$$k = \sqrt{2\mu E/\hbar^2}$$

$$\mathbf{v} = \mathbf{p}/\mu = \hbar\mathbf{k}/\mu$$

$$j_i = v_i|A|^2$$

Scattered wave

$$\psi^{\text{scat}} = Af(\theta,\phi)\frac{\mathrm{e}^{\mathrm{i}k_f R}}{R}$$

Outgoing flux

$$j_f = v_f|A|^2|f(\theta,\phi)|^2/R^2$$

Asymptotic wave

$$\psi^{\text{asym}} = \psi^{\text{beam}} + \psi^{\text{scat}} = A\left[\mathrm{e}^{\mathrm{i}k_i z} + f(\theta,\phi)\frac{\mathrm{e}^{\mathrm{i}k_f R}}{R}\right]$$

Scattering amplitude

Scattering theory: scattering amplitude and xs

Scattered angular flux and incoming flux

$$\hat{j}_f = R^2 j_f = v_f |A|^2 |f(\theta,\phi)|^2$$

$$j_i = v_i |A|^2$$

$$k = \sqrt{2\mu E/\hbar^2}$$

$$\mathbf{v} = \mathbf{p}/\mu = \hbar\mathbf{k}/\mu$$

Cross section

$$\sigma(\theta,\phi) = \frac{v_f}{v_i} |f(\theta,\phi)|^2$$

Renormalized scattering amplitude

$$\tilde{f}(\theta,\phi) = \sqrt{\frac{v_f}{v_i}} f(\theta,\phi)$$

$$\sigma(\theta,\phi) = |\tilde{f}(\theta,\phi)|^2$$

Scattering equation: single channel central V

o short range potentials V(**R**)=0, $R>R_n$

no Coulomb for now

o positive energy time-independent Schrodinger eq to obtain $f(\theta,\phi)$

numerical solutions matched to asymptotic form

o spherical potentials V(**R**)=V(R)

angular momentum and energy commute

initial beam is cylindrically symm (m=0) implies scattered wave is too: $f(\theta,\phi)= f(\theta)$

$$\hat{T} = -\frac{\hbar^2}{2\mu}\nabla_R^2$$

$$= \frac{\hbar^2}{2\mu}\left[-\frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial}{\partial R}\right) + \frac{\hat{L}^2}{R^2}\right]$$

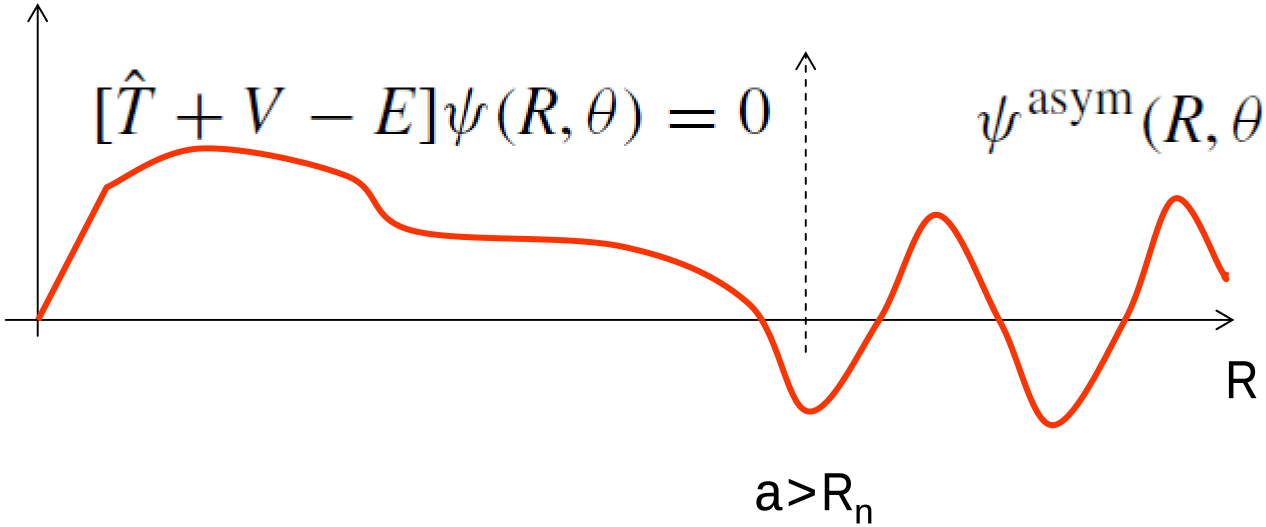
$$[\hat{T} + V - E]\psi(R,\theta) = 0$$

Solving scattering eq: overall scheme

o Solution of scattering equation needs to match onto asymptotic form

$$[\hat{T} + V - E]\psi(R,\theta) = 0$$

$$\psi^{\mathrm{asym}}(R,\theta) = e^{ikz} + f(\theta)\frac{e^{ikR}}{R}$$



Partial wave expansion

- Legendre polynomials form a complete set $\sum_L b_L(R) P_L(\cos\theta)$
- they are eigenstates of $\hat{L}^2$ and $\hat{L}_z$
- orthogonality relation: $\int_0^\pi P_L(\cos\theta) P_{L'}(\cos\theta) \sin\theta \mathrm{d}\theta = \frac{2}{2L+1}\delta_{LL'}$
- particular form for expansion

$$\nabla_R^2 P_L(\cos\theta) \frac{\chi_L(R)}{R} = \frac{1}{R}\left(\frac{\mathrm{d}^2}{\mathrm{d}R^2} - \frac{L(L+1)}{R^2}\right)\chi_L(R)\, P_L(\cos\theta)$$

- partial wave expansion:

$$\psi(R,\theta) = \sum_{L=0}^{\infty}(2L+1)\mathrm{i}^L P_L(\cos\theta)\frac{1}{kR}\chi_L(R)$$

- partial wave equation:

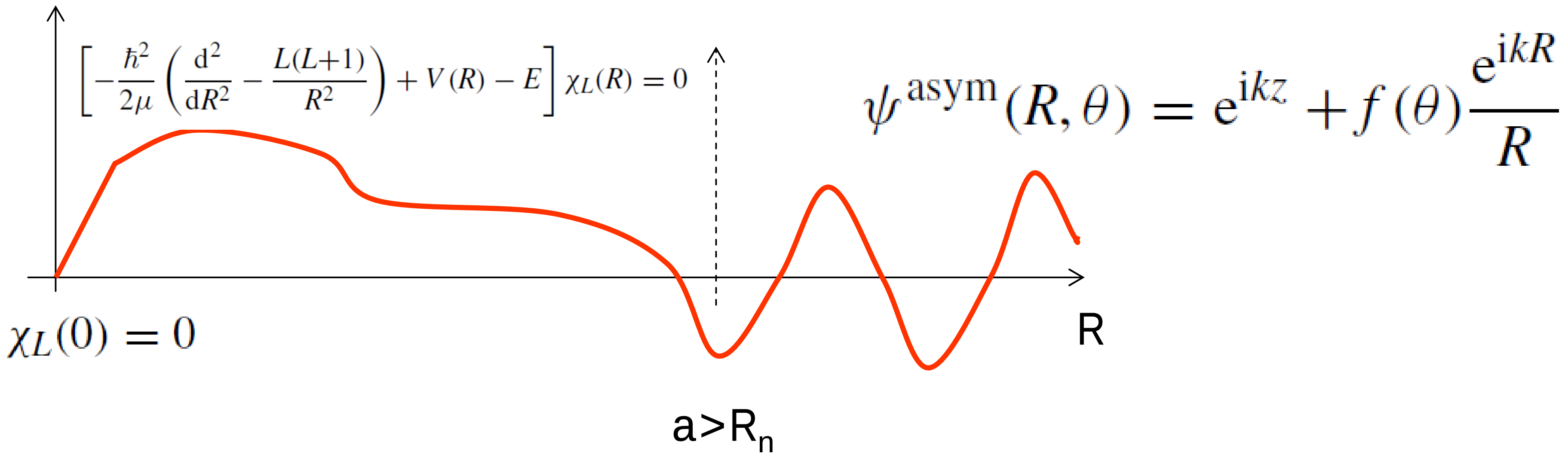
$$\left[-\frac{\hbar^2}{2\mu}\left(\frac{\mathrm{d}^2}{\mathrm{d}R^2} - \frac{L(L+1)}{R^2}\right) + V(R) - E\right]\chi_L(R) = 0$$

Matching to asymptotics

o 2nd order differential equation -> two boundary conditions

1. for wfn to be finite everywhere $\chi_L(0) = 0$
2. asymptotically

$$\psi^{\text{asym}}(R,\theta) = e^{ikz} + f(\theta)\frac{e^{ikR}}{R}$$



free solution and coulomb functions

o when V(**R**)=0, for all R

$$\rho \equiv kR$$

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}\rho^2} - \frac{L(L+1)}{\rho^2} + 1\right]\chi_L^{\mathrm{ext}}(\rho/k) = 0$$

$$\eta = 0$$

o Coulomb wave equation

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}\rho^2} - \frac{L(L+1)}{\rho^2} - \frac{2\eta}{\rho} + 1\right]X_L(\eta,\rho) = 0$$

Sommerfeld parameter

$$\eta = \frac{Z_1 Z_2 e^2}{\hbar v} = \frac{Z_1 Z_2 e^2 \mu}{\hbar^2 k} = \frac{Z_1 Z_2 e^2}{\hbar}\left(\frac{\mu}{2E}\right)^{\frac{1}{2}}$$

o two linearly independent solutions:

regular and irregular Coulomb functions

$$F_L(\eta,\rho) \quad G_L(\eta,\rho)$$

o two linearly independent solutions:

outgoing and incoming Hanckel functions

$$H_L^{\pm} = G_L \pm \mathrm{i}F_L$$

Others may use different notation: $H^- = I$ and $H^+ = O$

Properties of F/G/H with η=0 (Bessel functions)

The $\eta = 0$ functions are more directly known in terms of Bessel functions:

$$F_L(0,\rho) = \rho\, j_L(\rho) = (\pi\rho/2)^{1/2} J_{L+1/2}(\rho)$$

$$G_L(0,\rho) = -\rho\, y_L(\rho) = -(\pi\rho/2)^{1/2} Y_{L+1/2}(\rho),$$

Their behaviour near the origin, for $\rho \ll L$, is

$$F_L(0,\rho) \sim \frac{1}{(2L+1)(2L-1)\cdots 3.1}\rho^{L+1}$$

$$G_L(0,\rho) \sim (2L-1)\cdots 3.1\ \rho^{-L},$$

and their asymptotic behaviour when $\rho \gg L$ is

$$F_L(0,\rho) \sim \sin(\rho - L\pi/2)$$

$$G_L(0,\rho) \sim \cos(\rho - L\pi/2)$$

$$H_L^{\pm}(0,\rho) \sim \mathrm{e}^{\pm\mathrm{i}(\rho - L\pi/2)} = \mathrm{i}^{\mp L}\mathrm{e}^{\pm\mathrm{i}\rho}.$$

So H_L^+ describes an outgoing wave $\mathrm{e}^{\mathrm{i}\rho}$, and H_L^- an incoming wave $\mathrm{e}^{-\mathrm{i}\rho}$.

Partial wave expansion for plane wave

$$\psi(R,\theta) = \sum_{L=0}^{\infty}(2L+1)\mathrm{i}^L P_L(\cos\theta)\frac{1}{kR}\chi_L(R)$$

$$\mathrm{e}^{\mathrm{i}kz} = \sum_{L=0}^{\infty}(2L+1)\mathrm{i}^L P_L(\cos\theta)\frac{1}{kR}F_L(0,kR)$$

$$\mathrm{e}^{\mathrm{i}kz} = \sum_{L=0}^{\infty}(2L+1)\mathrm{i}^L P_L(\cos\theta)\frac{1}{kR}\frac{\mathrm{i}}{2}[H_L^-(0,kR) - H_L^+(0,kR)]$$

incoming outgoing

o at large distances the radial wavefunction should behave as

$$\chi_L^{\mathrm{ext}}(R) = A_L\left[H_L^-(0,kR) - \mathbf{S}_L H_L^+(0,kR)\right]$$

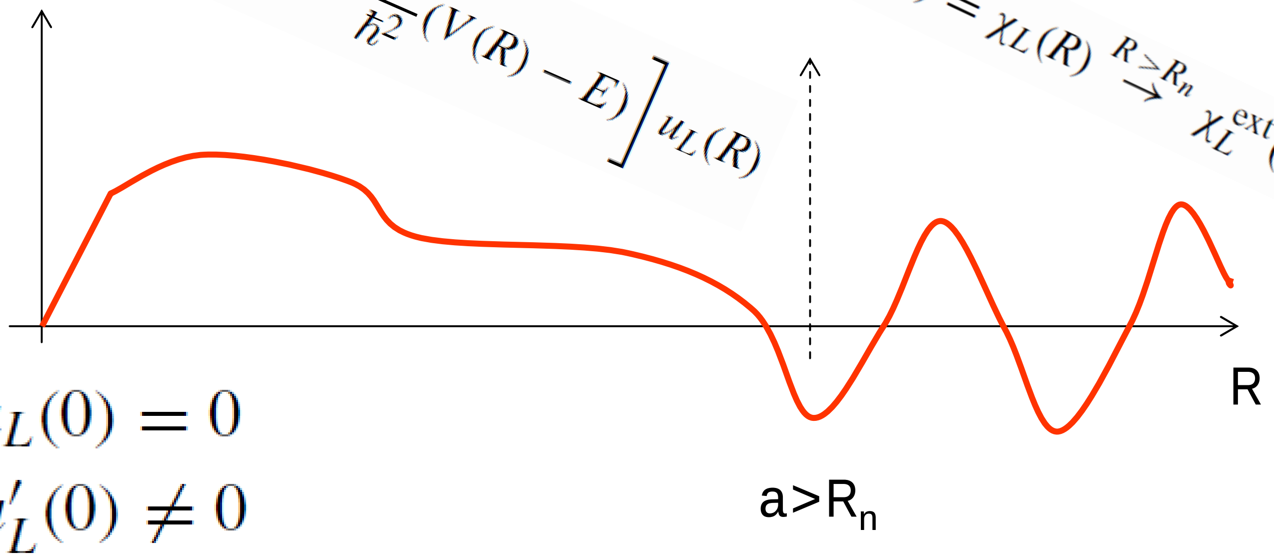
partial wave S-matrix element

Matching to asymptotics

o numerical solution is proportional to true solution $\chi_L(R) = B u_L(R)$

$$u_L''(R) = \left[\frac{L(L+1)}{R^2} + \frac{2\mu}{\hbar^2}(V(R)-E)\right] u_L(R)$$

$$B u_L(R) = \chi_L(R) \overset{R>R_n}{\rightarrow} \chi_L^{\mathrm{ext}}(R) = A_L\left[H_L^-(0,kR) - \mathbf{S}_L H_L^+(0,kR)\right]$$



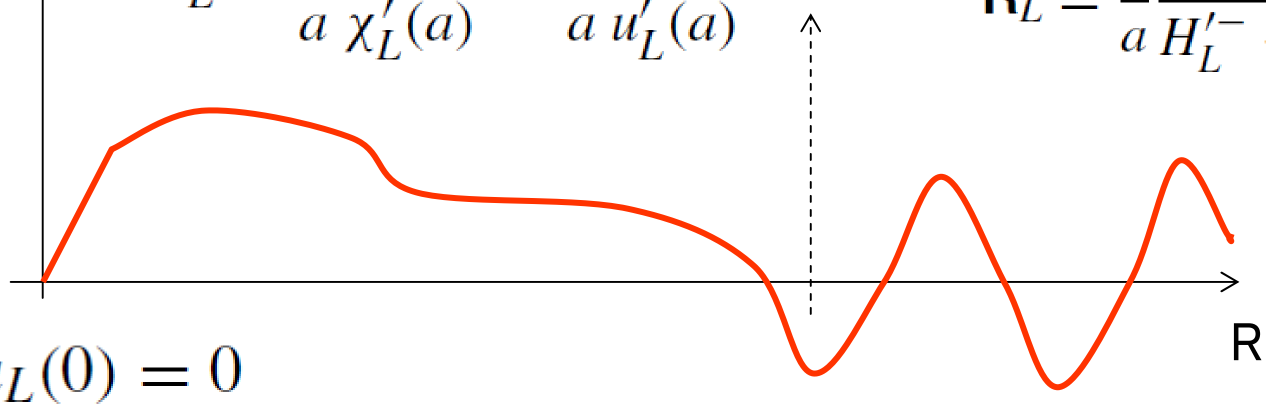
$$u_L(0) = 0$$
$$u_L'(0) \neq 0$$

Inverse logarithmic derivative

- The matching can be done with the inverse log derivative R_L
- any potential will produce R_L which relates to S_L

$$\mathbf{R}_L = \frac{1}{a}\frac{\chi_L(a)}{\chi'_L(a)} = \frac{1}{a}\frac{u_L(a)}{u'_L(a)}$$

$$\mathbf{R}_L = \frac{1}{a}\frac{H_L^- - \mathbf{S}_L H_L^+}{H_L'^- - \mathbf{S}_L H_L'^+}$$



$$u_L(0) = 0$$

$$u'_L(0) \neq 0$$

$a > R_n$

$$\mathbf{S}_L = \frac{H_L^- - a\mathbf{R}_L H_L'^-}{H_L^+ - a\mathbf{R}_L H_L'^+}$$

S-matrix and scattering amplitude

NSCL

o to obtain the scattering amplitude need to sum the partial waves

$$\psi(R,\theta) \stackrel{R>R_n}{\rightarrow} \frac{1}{kR}\sum_{L=0}^{\infty}(2L+1)\mathrm{i}^L P_L(\cos\theta) A_L [H_L^-(0,kR) - \mathbf{S}_L H_L^+(0,kR)]$$

$$\psi^{\mathrm{asym}}(R,\theta) = \mathrm{e}^{\mathrm{i}kz} + f(\theta)\frac{\mathrm{e}^{\mathrm{i}kR}}{R}$$

o 1) derive the relations below

$$f(\theta) = \frac{1}{2\mathrm{i}k}\sum_{L=0}^{\infty}(2L+1)P_L(\cos\theta)(\mathbf{S}_L - 1)$$

$$\sigma(\theta) \equiv \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \left|\frac{1}{2\mathrm{i}k}\sum_{L=0}^{\infty}(2L+1)P_L(\cos\theta)(\mathbf{S}_L - 1)\right|^2$$

Phase shifts

o Each partial wave S-matrix can be equivalently described with a phase shift

$$\mathbf{S}_L = e^{2i\delta_L}$$

$$\delta_L(E) = \frac{1}{2i} \ln \mathbf{S}_L + n(E)\pi$$

added to make the phase shift continuous

o scattering amplitude in terms of phase shifts

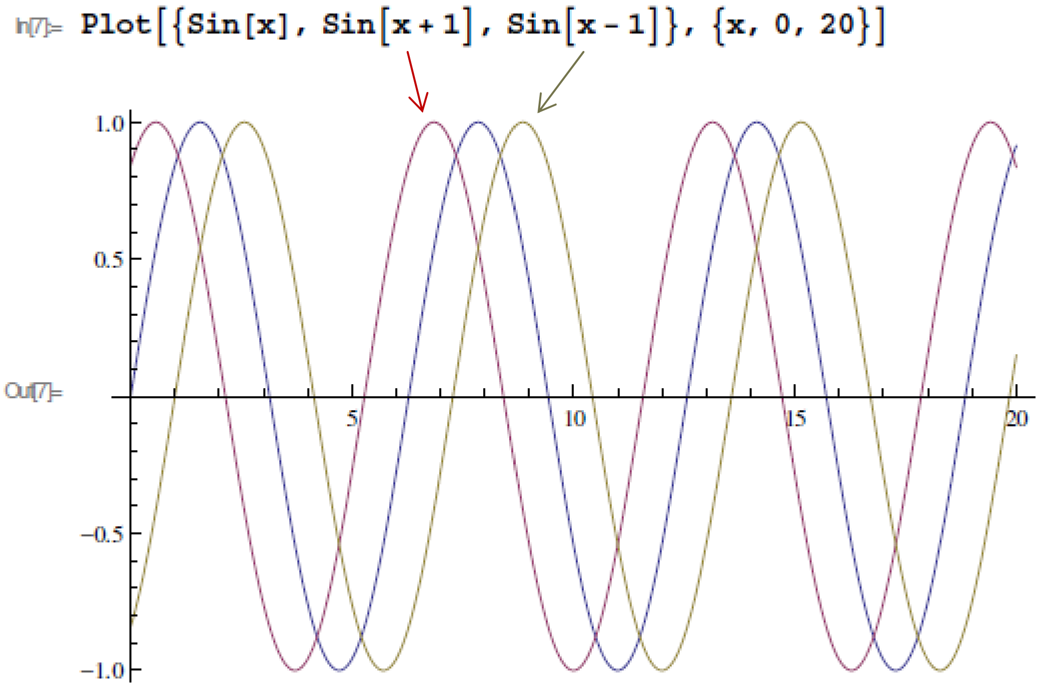
$$f(\theta) = \frac{1}{k} \sum_{L=0}^{\infty} (2L+1) P_L(\cos\theta) e^{i\delta_L} \sin\delta_L$$

o asymptotic form in terms of phase shift

$$\chi_L^{\text{ext}}(R) \to e^{i\delta_L}[\cos\delta_L \sin(kR - L\pi/2) + \sin\delta_L \cos(kR - L\pi/2)]$$
$$= e^{i\delta_L} \sin(kR + \delta_L - L\pi/2).$$

Phase shifts as a function of energy

o attractive potentials: delta>0
o repulsive potentials: delta<0



T-matrix

o the partial wave T-matrix is defined as the amplitude of the outgoing wave

$$\chi_L^{\mathrm{ext}}(R) = F_L(0,kR) + \mathbf{T}_L H_L^+(0,kR) \qquad \mathbf{S}_L = 1 + 2\mathrm{i}\mathbf{T}_L$$

o simple relation with the scattering amplitude

$$f(\theta) = \frac{1}{k}\sum_{L=0}^{\infty}(2L+1)P_L(\cos\theta)\mathbf{T}_L$$

Relations between T, S, δ

Using:	δ	**K**	**T**	**S**
$\chi(R) =$	$e^{i\delta}[F\cos\delta + G\sin\delta]$	$\frac{1}{1-i\mathbf{K}}[F + \mathbf{K}G]$	$F + \mathbf{T}H^{+}$	$\frac{i}{2}[H^{-} - \mathbf{S}H^{+}]$
$\delta =$	δ	$\arctan \mathbf{K}$	$\arctan \frac{\mathbf{T}}{1+i\mathbf{T}}$	$\frac{1}{2i}\ln \mathbf{S}$
$\mathbf{K} =$	$\tan\delta$	$\mathbf{K}$	$\frac{\mathbf{T}}{1+i\mathbf{T}}$	$i\frac{1-\mathbf{S}}{1+\mathbf{S}}$
$\mathbf{T} =$	$e^{i\delta}\sin\delta$	$\frac{\mathbf{K}}{1-i\mathbf{K}}$	$\mathbf{T}$	$\frac{i}{2}(1-\mathbf{S})$
$\mathbf{S} =$	$e^{2i\delta}$	$\frac{1+i\mathbf{K}}{1-i\mathbf{K}}$	$1+2i\mathbf{T}$	$\mathbf{S}$
$V = 0$	$\delta = 0$	$\mathbf{K} = 0$	$\mathbf{T} = 0$	$\mathbf{S} = 1$
V real	δ real	$\mathbf{K}$ real	$\lvert 1+2i\mathbf{T}\rvert = 1$	$\lvert\mathbf{S}\rvert = 1$

Integrated cross sections

o use properties of legendre polynomials

$$f(\theta) = \frac{1}{2ik}\sum_{L=0}^{\infty}(2L+1)P_L(\cos\theta)(\mathbf{S}_L - 1)$$

$$\begin{aligned}\sigma_{\rm el} &= \int_0^{2\pi} d\phi \int_0^{\pi} {\rm d}\theta \sin\theta \, \sigma(\theta) \\ &= 2\pi \int_0^{\pi} {\rm d}\theta \sin\theta |f(\theta)|^2 \\ &= \frac{\pi}{k^2}\sum_{L=0}^{\infty}(2L+1)|1-\mathbf{S}_L|^2 \\ &= \frac{4\pi}{k^2}\sum_{L=0}^{\infty}(2L+1)\sin^2\delta_L,\end{aligned}$$

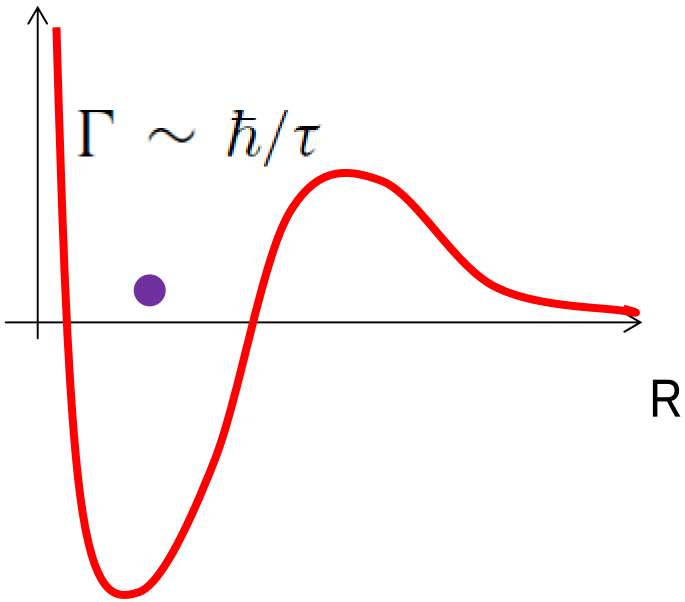
oOptical theorem:
total elastic cross section related to zero-angle scattering amplitude

$$f(0) = \frac{1}{2ik}\sum_{L=0}^{\infty}(2L+1)(e^{2i\delta_L} - 1),$$

$$\begin{aligned}{\rm Im} f(0) &= \frac{1}{k}\sum_{L=0}^{\infty}(2L+1)\sin^2\delta_L \\ &= \frac{k}{4\pi}\sigma_{\rm el}.\end{aligned}$$

Resonances and phase shifts

o particles trapped inside a barrier



oResonance characterized by J, E, $\Gamma>0$

o will show rapid rise of phase shift

$$\Delta t \sim \hbar \mathrm{d}\delta(E)/\mathrm{d}E$$

o there is usually a background in addition to the resonance part:

$$\delta(E) = \delta_{\mathrm{bg}}(E) + \delta_{\mathrm{res}}(E)$$

$$\delta_{\mathrm{res}}(E) = \arctan\left(\frac{\Gamma/2}{E_{\mathrm{r}} - E}\right) + n(E)\pi$$

o in a pure case, with no background at the resonance energy $\delta=\pi/2$

Resonances and cross sections

o Breit-Wigner form

$$\sigma_{\rm el}^{\rm res}(E) \simeq \frac{4\pi}{k^2}(2L+1)\sin^2\delta_{\rm res}(E)$$

$$= \frac{4\pi}{k^2}(2L+1)\frac{\Gamma^2/4}{(E-E_{\rm r})^2+\Gamma^2/4}$$

Resonances and S-matrix

o S-matrix form around the resonance

$$\mathbf{S}(E) = e^{2i\delta_{bg}(E)} \frac{E - E_r - i\Gamma/2}{E - E_r + i\Gamma/2}$$

o if analytic continuation to complex energies

S-matrix **pole** at $\mathbf{E_p = E_r - i\,\Gamma / 2}$

Resonances and cross sections

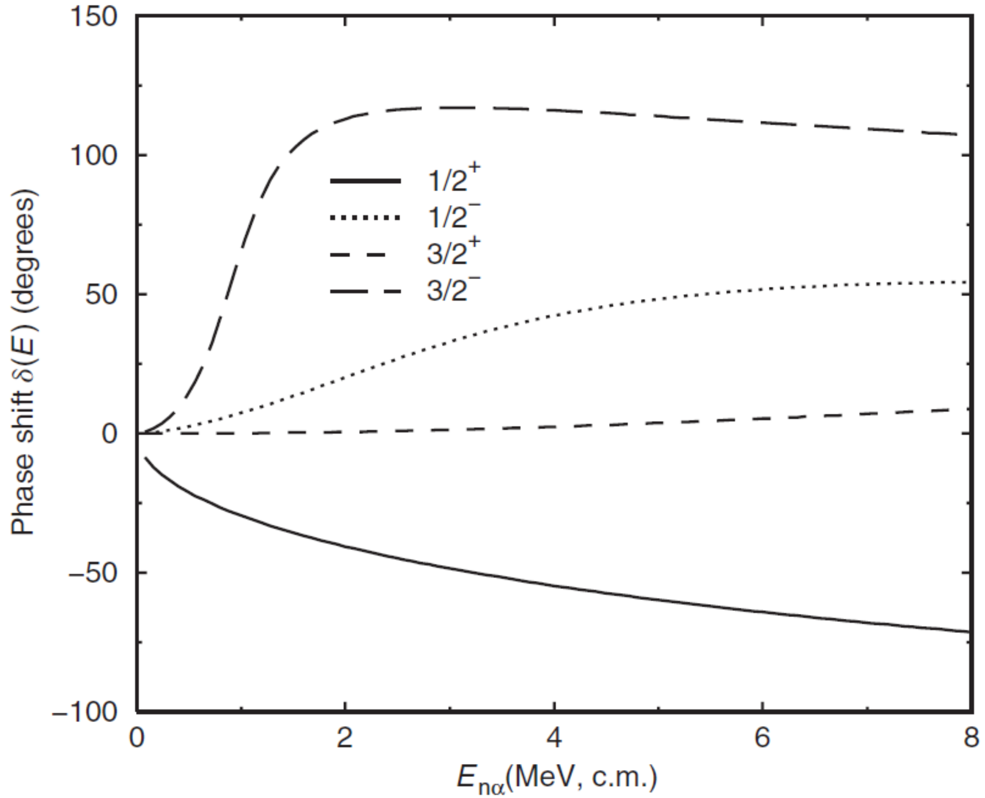


Fig. 3.2. Examples of resonant phase shifts for the $J^\pi = 3/2^-$ channel in low-energy n–α scattering, with a pole at $E = 0.96 - \text{i}0.92/2$ MeV. There is only a hint of a resonance in the phase shifts for the $J^\pi = 1/2^-$ channel, but it does have a wide resonant pole at $1.9 - \text{i}6.1/2$ MeV.

resonance signals

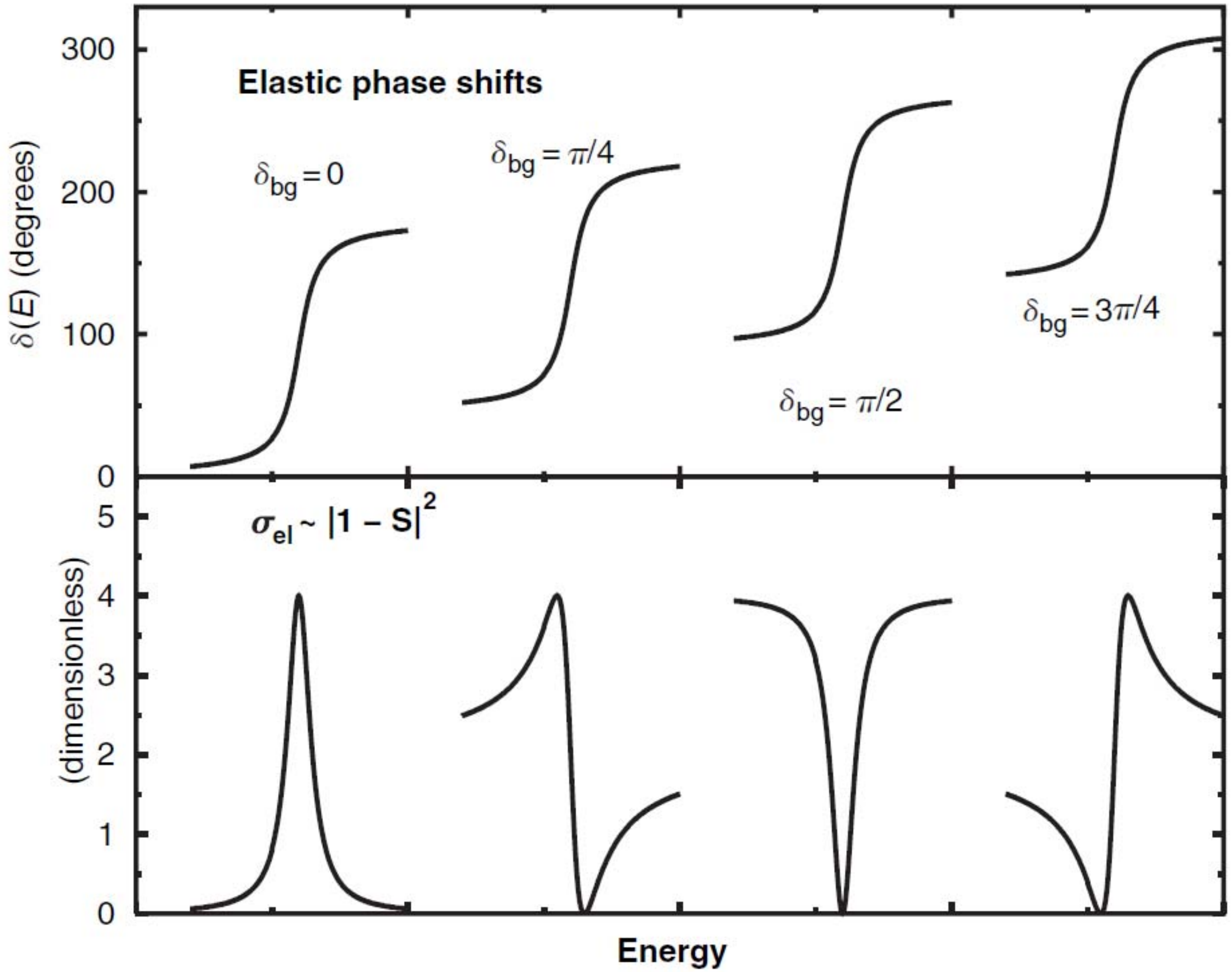


Fig. 3.3. Possible Breit-Wigner resonances. The upper panel shows resonant phase shifts with several background phase shifts $\delta_{bg} = 0, \pi/4, \pi/2$ and $3\pi/4$ in the same partial wave. The lower panel gives the corresponding contributions to the total elastic scattering cross section from that partial wave.

Complex energy plane

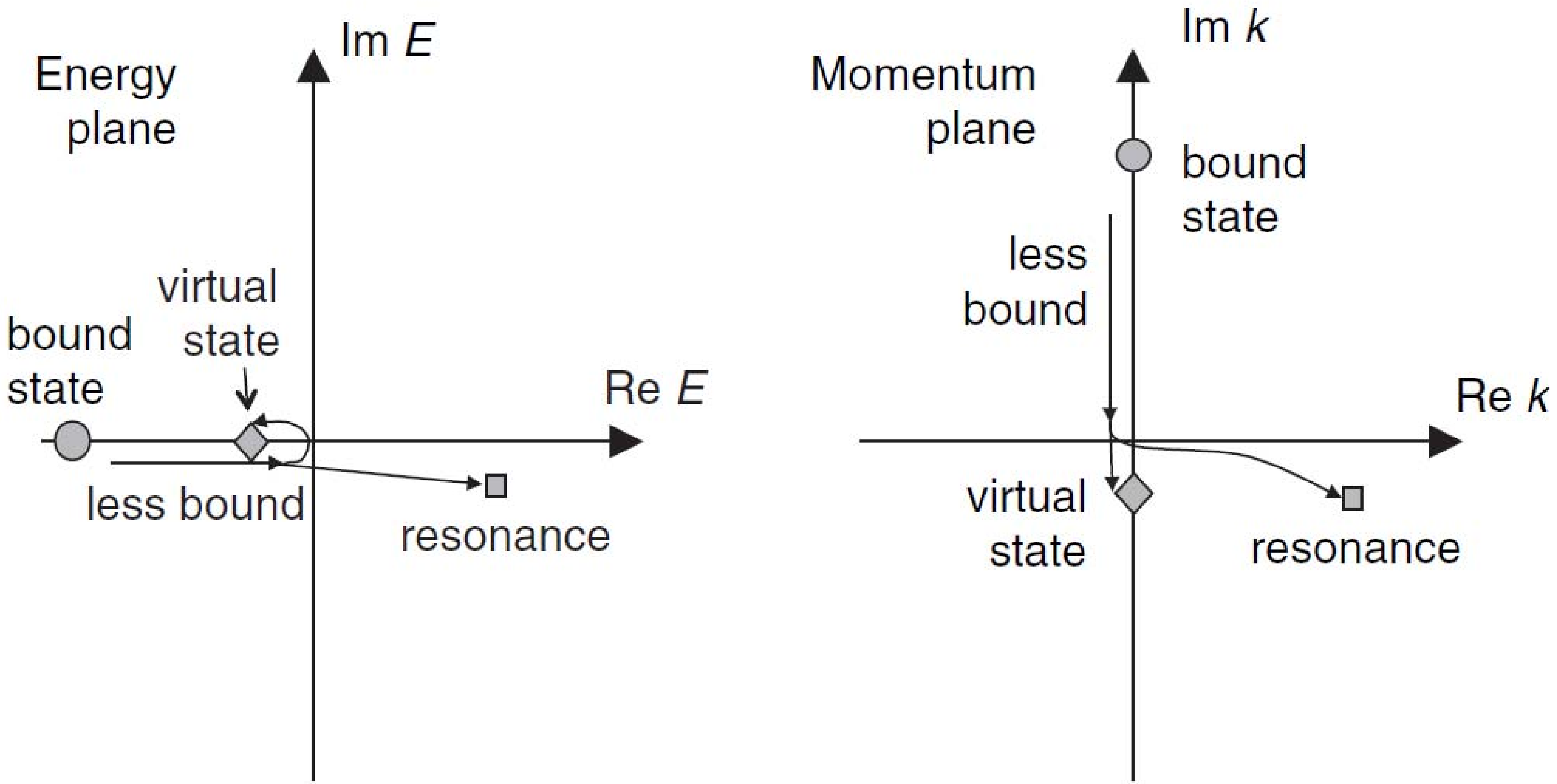


Fig. 3.4. The correspondences between the energy (left) and momentum (right) complex planes. The arrows show the trajectory of a bound state caused by a progressively weaker potential: it becomes a resonance for $L > 0$ or when there is a Coulomb barrier, otherwise it becomes a virtual state. Because $E \propto k^2$, bound states on the positive imaginary k axis and virtual states on the negative imaginary axis both map onto the negative energy axis.

Virtual states

o neutral L=0 particles: no barrier

o S-matrix pole is on negative imaginary k-axis (not a bound state!)

$$k_p = \pm\sqrt{2\mu E_p/\hbar^2}$$

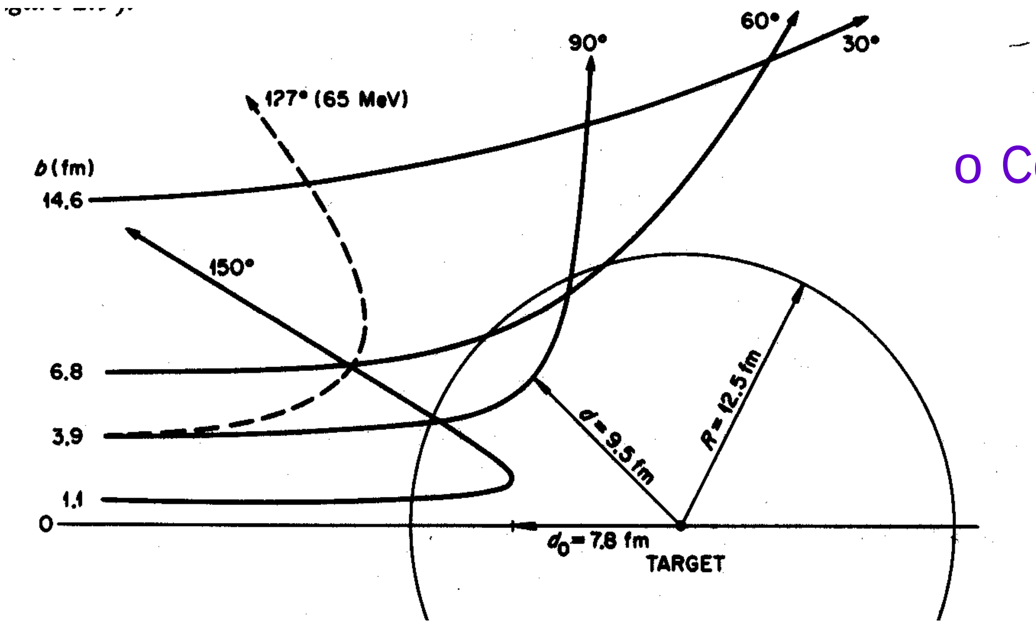
o scattering length $k_p = \mathrm{i}/a_0$

o S-matrix in terms of scattering length
$$\mathbf{S}(k) = -\frac{k + \mathrm{i}/a_0}{k - \mathrm{i}/a_0}$$

o phase shift in terms of scattering length

$$k \cot\delta(k) = -1/a_0$$

Classical Coulomb scattering

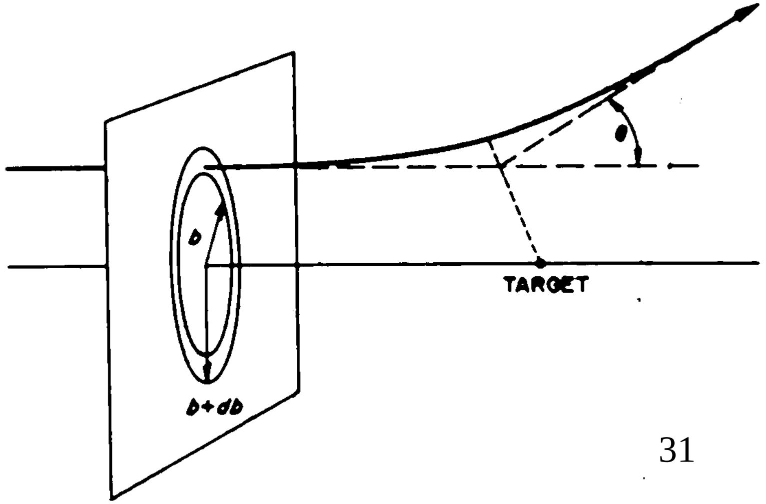


- Coulomb trajectories are hyperbolas

$$\tan\frac{\theta}{2} = \frac{\eta}{bk}$$

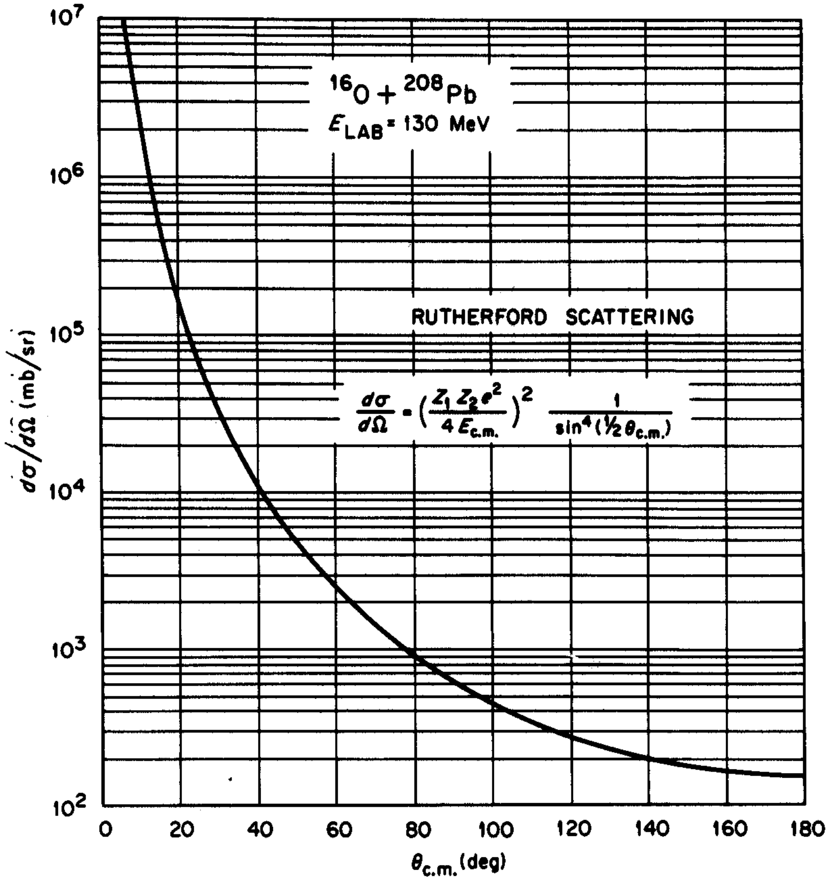
- the cross section for a pure Coulomb interaction is

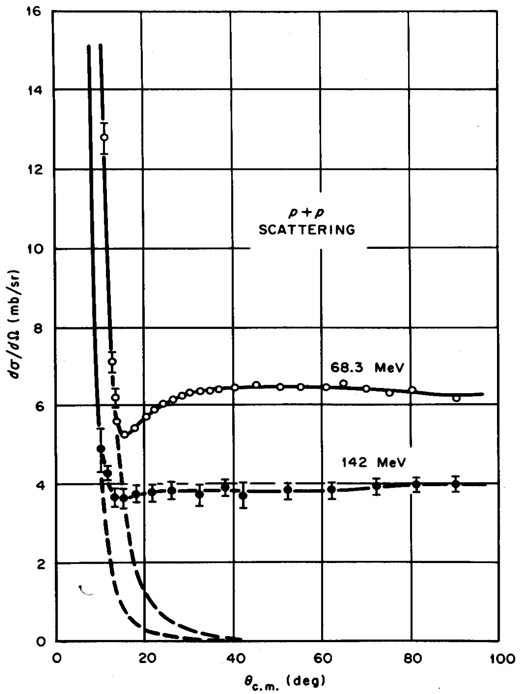
$$\sigma(\theta) \equiv \frac{b(\theta)}{\sin\theta}\frac{\mathrm{d}b}{\mathrm{d}\theta} = \frac{\eta^2}{4k^2\sin^4(\theta/2)}$$



Coulomb scattering

oexamples





Coulomb functions

o Coulomb wave equation

$$\left[\frac{d^2}{d\rho^2} - \frac{L(L+1)}{\rho^2} - \frac{2\eta}{\rho} + 1\right] X_L(\eta,\rho) = 0$$

$$F_L(\eta,\rho) = C_L(\eta)\rho^{L+1} e^{\mp i\rho}\ {}_1F_1(L+1 \mp i\eta; 2L+2; \pm 2i\rho)$$

$$C_L(\eta) = \frac{2^L e^{-\pi\eta/2}|\Gamma(1+L+i\eta)|}{(2L+1)!}$$

$${}_1F_1(a;b;z) = 1 + \frac{a}{b}\frac{z}{1!} + \frac{a(a+1)}{b(b+1)}\frac{z^2}{2!} + \frac{a(a+1)(a+2)}{b(b+1)(b+2)}\frac{z^3}{3!} + \cdots .$$

$$\begin{aligned} H_L^{\pm}(\eta,\rho) &= G_L(\eta,\rho) \pm iF_L(\eta,\rho) \\ &= e^{\pm i\Theta}(\mp 2i\rho)^{1+L\pm i\eta} U(1+L \pm i\eta, 2L+2, \mp 2i\rho) \end{aligned}$$

$$\Theta \equiv \rho - L\pi/2 + \sigma_L(\eta) - \eta \ln(2\rho)$$

$$\sigma_L(\eta) = \arg\Gamma(1+L+i\eta)$$

Coulomb functions

oBehaviour near the origin

$$F_L(\eta,\rho) \sim C_L(\eta)\rho^{L+1}, \quad G_L(\eta,\rho) \sim \left[(2L+1)C_L(\eta)\,\rho^L\right]^{-1}$$

$$C_0(\eta) = \sqrt{\frac{2\pi\eta}{e^{2\pi\eta}-1}} \quad \text{and} \quad C_L(\eta) = \frac{\sqrt{L^2+\eta^2}}{L(2L+1)} C_{L-1}(\eta)$$

A transition from small-ρ power law behavior to large-ρ oscillatory behavior occurs outside the classical turning point. This point is where $1 = 2\eta/\rho + L(L+1)/\rho^2$, namely

$$\rho_{tp} = \eta \pm \sqrt{\eta^2 + L(L+1)}. \qquad (3.1.67)$$

oBehaviour at large distances

$$F_L(\eta,\rho) \sim \sin\Theta, \quad G_L(\eta,\rho) \sim \cos\Theta, \quad \text{and} \quad H_L^{\pm}(\eta,\rho) \sim e^{\pm i\Theta}$$

$$\Theta \equiv \rho - L\pi/2 + \sigma_L(\eta) - \eta\ln(2\rho)$$

Coulomb scattering – partial wave

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- pure Coulomb Schrodinger eq can be solved exactly:

$$V_c(R) = Z_1 Z_2 e^2 / R$$

$$\psi_c(\mathbf{k}, \mathbf{R}) = \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{R}}\, \mathrm{e}^{-\pi\eta/2}\, \Gamma(1+\mathrm{i}\eta)\; {}_1F_1(-\mathrm{i}\eta; 1; \mathrm{i}(kR - \mathbf{k}\cdot\mathbf{R}))$$

- generalize the partial wave form of the plane wave

$$\psi_c(k\hat{\mathbf{z}}, \mathbf{R}) = \sum_{L=0}^{\infty} (2L+1)\mathrm{i}^L P_L(\cos\theta) \frac{1}{kR} F_L(\eta, kR)$$

- asymptotic form of the scattering wavefunction

$$\psi_c(k\hat{\mathbf{z}}, \mathbf{R}) \overset{R-Z\to\infty}{\rightarrow} \mathrm{e}^{\mathrm{i}[kz+\eta \ln k(R-z)]} + f_c(\theta) \frac{\mathrm{e}^{\mathrm{i}[kR-\eta \ln 2kR]}}{R}$$

Coulomb scattering amplitude

o formally can be written in partial wave expansion

$$f_c(\theta) = \frac{1}{2ik} \sum_{L=0}^{\infty} (2L+1) P_L(\cos\theta)(e^{2i\sigma_L(\eta)} - 1)$$

o series does not converge!

o without partial wave expansion one can derive the scattering amplitude

$$f_c(\theta) = -\frac{\eta}{2k\sin^2(\theta/2)} \exp\left[-i\eta\ln(\sin^2(\theta/2)) + 2i\sigma_0(\eta)\right]$$

Homework: 1) Derive this expression

Point-Coulomb cross section

$$\sigma_{\mathrm{Ruth}}(\theta) = |f_c(\theta)|^2 = \frac{\eta^2}{4k^2\sin^4(\theta/2)}$$

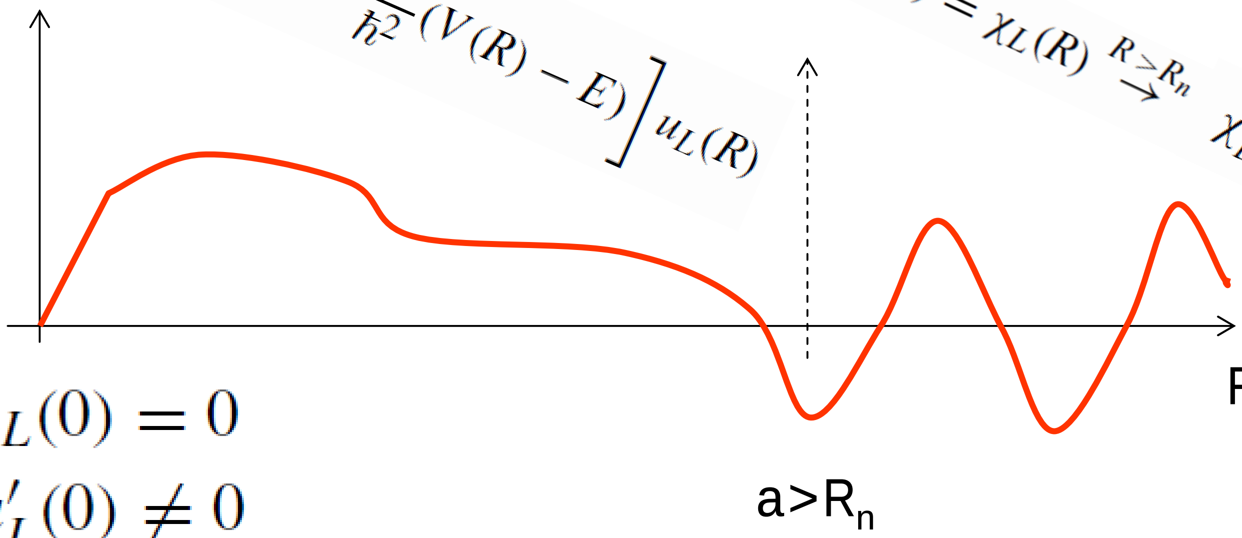
Generalized scattering problem w Coulomb

o numerical solution is proportional to true solution $\chi_L(R) = Bu_L(R)$

$$u_L''(R) = \left[\frac{L(L+1)}{R^2} + \frac{2\mu}{\hbar^2}(V(R)-E)\right]u_L(R)$$

Coulomb+nuclear

$$Bu_L(R) = \chi_L(R) \underset{R>R_n}{\rightarrow} \chi_L^{ext}(R) = \frac{i}{2}[H_L^-(\eta,kR) - \mathbf{S}_L^n H_L^+(\eta,kR)]$$



$$u_L(0) = 0$$
$$u_L'(0) \neq 0$$

Coulomb+nuclear

o generalized asymptotic form defines the nuclear S-matrix

$$\chi_L^{\text{ext}}(R) = \frac{\mathrm{i}}{2}[H_L^-(\eta, kR) - \mathbf{S}_L^n H_L^+(\eta, kR)]$$

o can be written in terms of the nuclear phase shift

$$\mathbf{S}_L^n = \mathrm{e}^{2\mathrm{i}\delta_L^n}$$

$$\chi_L^{\text{ext}}(R) = \mathrm{e}^{\mathrm{i}\delta_L^n}\left[\cos\delta_L^n\, F_L(\eta, kR) + \sin\delta_L^n\, G_L(\eta, kR)\right]$$

o combined phase shift from Coulomb and nuclear

$$\delta_L = \sigma_L(\eta) + \delta_L^n$$

Coulomb+nuclear

$$\delta_L = \sigma_L(\eta) + \delta_L^n$$

Coulomb + nuclear phase shifts

$$e^{2i\delta_L} - 1 = (e^{2i\sigma_L(\eta)} - 1) + e^{2i\sigma_L(\eta)}(e^{2i\delta_L^n} - 1)$$

$$f_{nc}(\theta) = f_c(\theta) + f_n(\theta)$$

$$f_n(\theta) = \frac{1}{2ik}\sum_{L=0}^{\infty}(2L+1)P_L(\cos\theta)e^{2i\sigma_L(\eta)}(\mathbf{S}_L^n - 1)$$

Coulomb+nuclear scattering

$$\sigma_{nc}(\theta) = |f_c(\theta) + f_n(\theta)|^2 \equiv |f_{nc}(\theta)|^2$$

$$\sigma/\sigma_{\rm Ruth} \equiv \sigma_{nc}(\theta)/\sigma_{\rm Ruth}(\theta)$$

Don’t add nuclear only and Coulomb only cross sections!

Optical potential

o Where does the optical potential come from?

Consider the original many-body problem nucleons-nucleus N+A

$$H(\mathbf{r}_0;\mathbf{r}_1,\mathbf{r}_2,\ldots,\mathbf{r}_A)\Psi(\mathbf{r}_0;\mathbf{r}_1,\mathbf{r}_2,\ldots,\mathbf{r}_A) = E\Psi(\mathbf{r}_0;\mathbf{r}_1,\mathbf{r}_2,\ldots,\mathbf{r}_A)$$

Split the Hamiltonian into:

o kinetic energy of the projectile

o the interaction of the projectile with all nucleons of the target

o internal Hamiltonian of the target

$$H(\mathbf{r}_0;\mathbf{r}_1,\mathbf{r}_2,\ldots,\mathbf{r}_A) = T_0 + \sum_{i=1}^{A} V(\mathbf{r}_{0i}) + H_A(\mathbf{r}_1,\mathbf{r}_2,\ldots,\mathbf{r}_A)$$

The solutions for the target Hamiltonian form a complete set:

$$H_A(\mathbf{r}_1,\mathbf{r}_2,\ldots,\mathbf{r}_A)\Phi_i(\mathbf{r}_1,\mathbf{r}_2,\ldots,\mathbf{r}_A) = \epsilon_i\Phi_i(\mathbf{r}_1,\mathbf{r}_2,\ldots,\mathbf{r}_A)$$

The general solution for N+A can be written in terms of the complete set above:

$$\Psi(\mathbf{r}_0;\mathbf{r}_1,\mathbf{r}_2,\ldots,\mathbf{r}_A) = \sum_{ij} \chi_i(\mathbf{r}_0)\Phi_j(\mathbf{r}_1,\mathbf{r}_2,\ldots,\mathbf{r}_A)$$

Wong, Introduction to Nuclear Physics, Wiley

Optical potential

o Feshbach projection

Since at this point we still assume in our reaction model that the target stays in the ground state, we need to project the problem into the target ground state.

P is the projection operator: $P = |\Phi_0\rangle\langle\Phi_0|$

It picks up the elastic component: $P\Psi = \chi_0\Phi_0$

Properties of projection operators

$$Q = 1 - P$$

$$P^2\Psi = P\Psi$$
$$Q^2\Psi = Q\Psi$$
$$PQ\Psi = QP\Psi = 0$$

Now apply it to the full equation: $(E - H)(P + Q)\Psi = 0$

Optical potential

NSCL

- After some algebra:

$$\left\{E - T_0 - \langle\Phi_0|V|\Phi_0\rangle - \langle\Phi_0|VQ\frac{1}{E-QHQ}QV|\Phi_0\rangle\right\}\chi_0 = 0$$

$$V \equiv \sum_{i=1}^{A} V(r_{0i})$$

Potential acting between projectile and target nucleons

$$H_A\Phi_0 = \epsilon_0\Phi_0 = 0$$

Interpretation for the formal propagator: multiple scattering in Q-space

$$\frac{1}{E-QHQ} = \frac{1}{E}\left\{1 + \frac{1}{E}QHQ + \frac{1}{E}QHQ\frac{1}{E}QHQ + \cdots\right\}$$

- The scattering equation can be rewritten: $(E - T_0 - \mathcal{V}(\boldsymbol{r}_0))\chi_0 = 0$

  with the effective potential: $\mathcal{V}(\boldsymbol{r}_0) = \langle\Phi_0|V|\Phi_0\rangle + \langle\Phi_0|VQ\frac{1}{E-QHQ}QV|\Phi_0\rangle$

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Optical potential

o The scattering equation can be rewritten: $(E - T_0 - \mathcal{V}(\boldsymbol{r}_0))\chi_0 = 0$

with the effective potential: $$\mathcal{V}(\boldsymbol{r}_0) = \langle\Phi_0|V|\Phi_0\rangle + \langle\Phi_0|VQ\frac{1}{E-QHQ}QV|\Phi_0\rangle$$

Homework: 2) derive this equation

oThis potential is generally non-local which gives rise to some complications:

$$(E - T_0)\chi_0(\boldsymbol{r}_0) = \mathcal{V}(\boldsymbol{r}_0)\chi_0(\boldsymbol{r}_0) + \int f(\boldsymbol{r}_0, \boldsymbol{r}_0')\chi_0(\boldsymbol{r}_0')d\boldsymbol{r}_0'$$

Often this is approximated to a local version.
The optical model replaces this microscopic potential by a model potential obtained phenomenologically: $(E - T_0 - U_{\text{opt}})\chi_0 = 0$

Scattering into Q-space may never return to elastic – loss of flux

Optical potential needs to have an imaginary term!

Wong, Introduction to Nuclear Physics, Wiley

Optical potential

oExample of a microscopically derived optical potential: folding

$$U_{\text{opt}}(\boldsymbol{r}_0) \approx \langle \Phi_0(\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_A) | \sum_{i=1}^{A} V(\boldsymbol{r}_{0i}) | \Phi_0(\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_A) \rangle$$

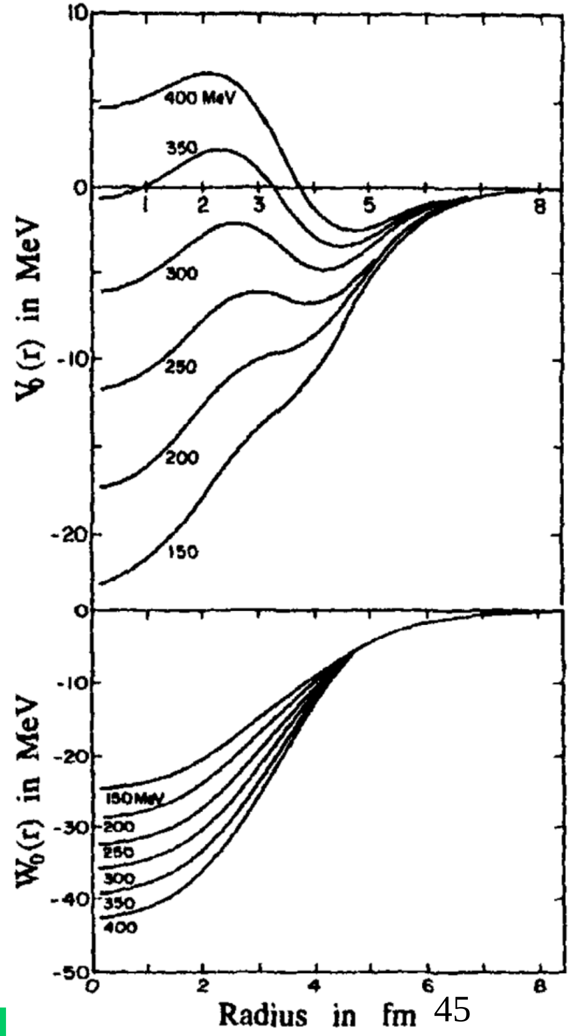
free or medium NN interaction?
density dep?

oIn principle antisymmetrization need to be included:

$$\langle \Phi_0 | V | \Phi_0 \rangle = \langle \Phi_0 | t_D | \Phi_0 \rangle + \langle \Phi_0 | t_E | \Phi_0 \rangle$$

direct — exchange

Radial shape of the volume term for p+A at different beam energies: folding using Paris potential



Wong, Introduction to Nuclear Physics, Wiley

Optical potential

In principle antisymmetrization need to be included:

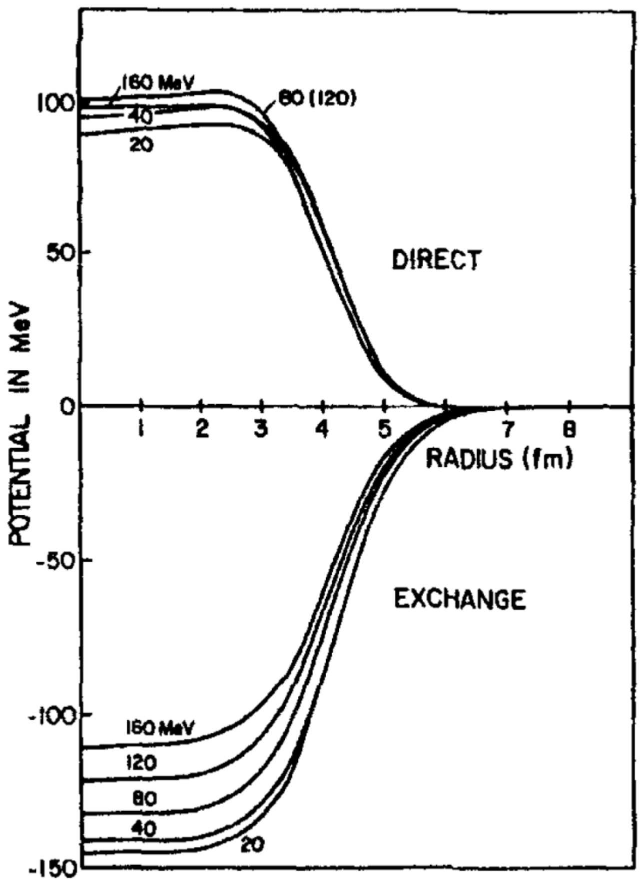
$$\langle\Phi_0|V|\Phi_0\rangle = \langle\Phi_0|t_D|\Phi_0\rangle + \langle\Phi_0|t_E|\Phi_0\rangle$$

Direct part depends of the density:

$$U^D_{\mathrm{opt}}(\boldsymbol{r}_0, E) = \int \rho(\boldsymbol{r}) t_D(\boldsymbol{r}_0, \boldsymbol{r}, \rho, E)\, d^3 r$$

$$\rho(\boldsymbol{r}) = |\Phi_0\rangle\langle\Phi_0| \approx \sum_{i=1}^{A} \phi_i^*(\boldsymbol{r})\phi_i(\boldsymbol{r})$$

The exchange part is non-local in general



Radial shape of the direct and exchange part for p+A optical potential at different beam energies: NN-Paris potential

Wong, Introduction to Nuclear Physics, Wiley

Optical potentials

o all the terms to be considered: $V_c(R) + V(R) + i\,W(R) + V_{so}(R)$

o loss of flux - absorption (W<0)

o Nucleon potentials as described with Woods-Saxon shape
(to mimic the density distribution in nuclei)

$$V(R) = -\frac{V_r}{1+\exp\left(\frac{R-R_r}{a_r}\right)}$$

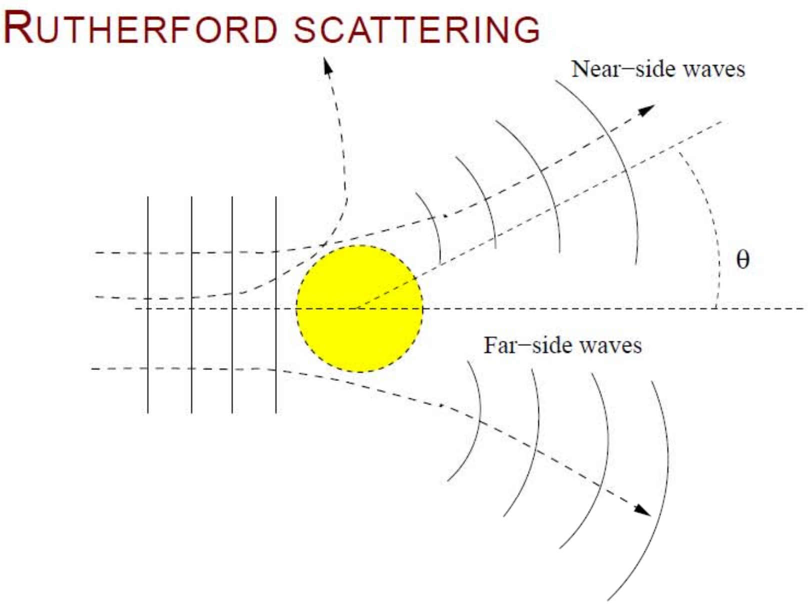
$$W(R) = -\frac{W_i}{1+\exp\left(\frac{R-R_i}{a_i}\right)}$$

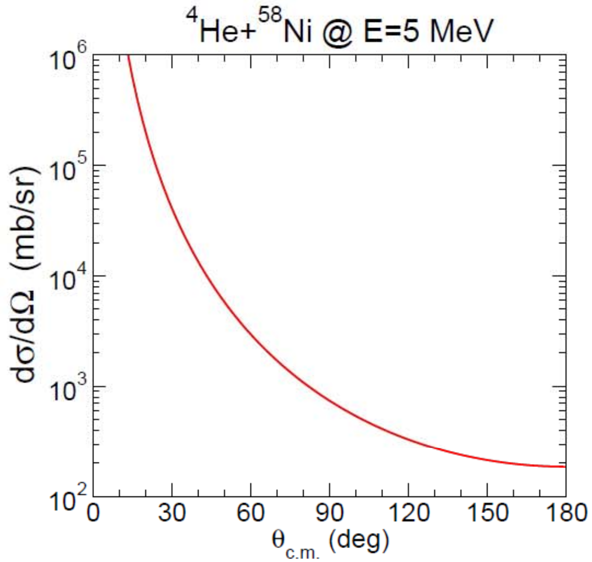
o Sometimes imaginary also defined at $d/dR(V_{ws}(r))$ - surface

$$R_r = r_r A^{1/3}$$

For nucleon interaction V=40-50 MeV, r=1.2 fm and a=0.6-0.65 fm

elastic scattering: examples



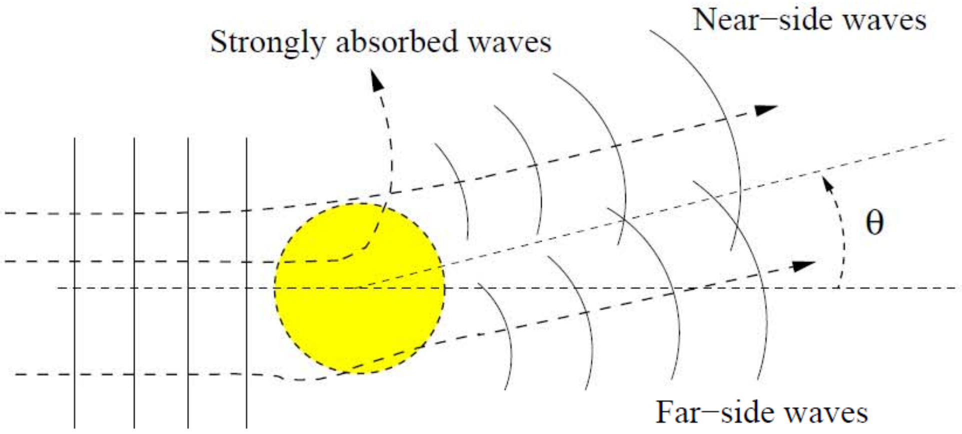


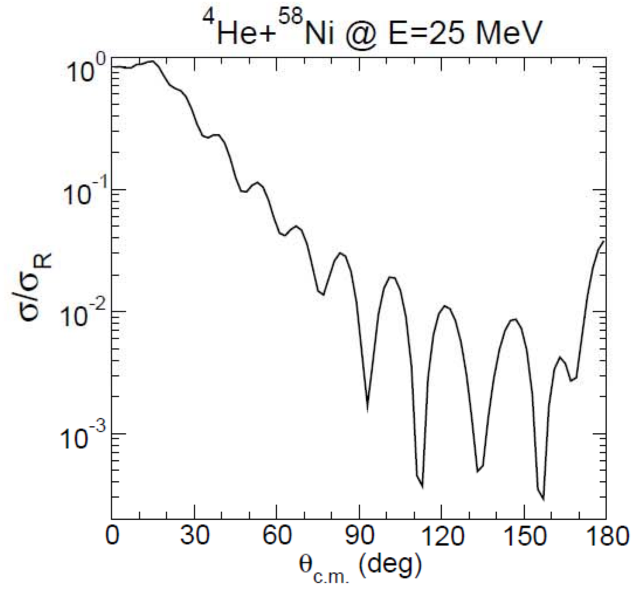
- Purely Coulomb potential ($\eta \ggg 1$)
- Bombarding energy well below the Coulomb barrier
- Obeys Rutherford law:

$$\frac{d\sigma}{d\Omega} = \frac{zZe^2}{4E} \frac{1}{\sin^4(\theta/2)}$$

elastic scattering: examples

FRAUNHOFER SCATTERING:

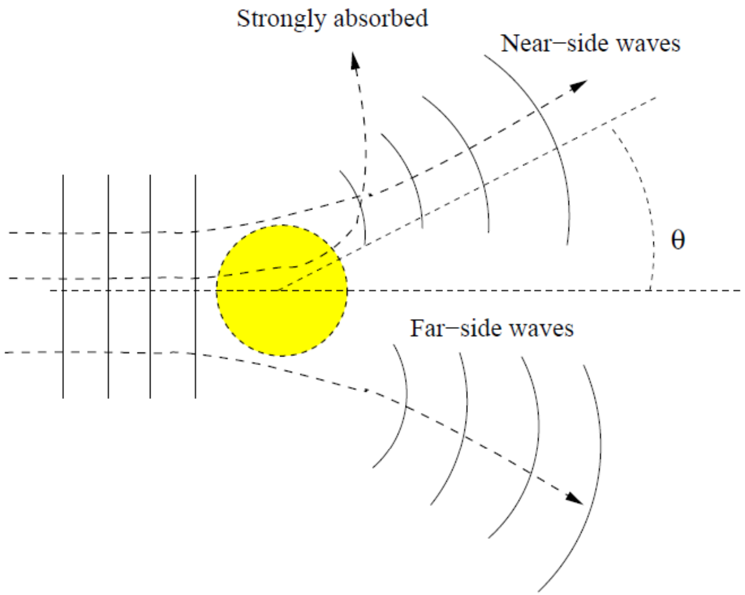


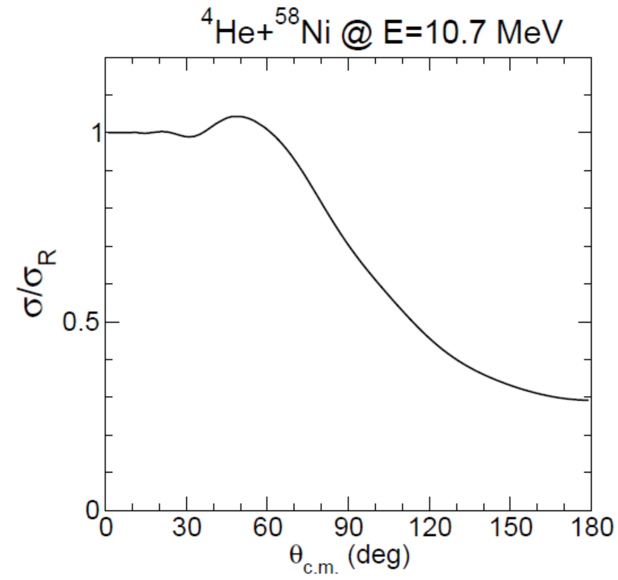


- Bombarding energy well above Coulomb barrier
- Coulomb weak ($\eta \lesssim 1$)
- Nearside/farside interference pattern (difracction)

elastic scattering: examples

FRESNEL SCATTERING:



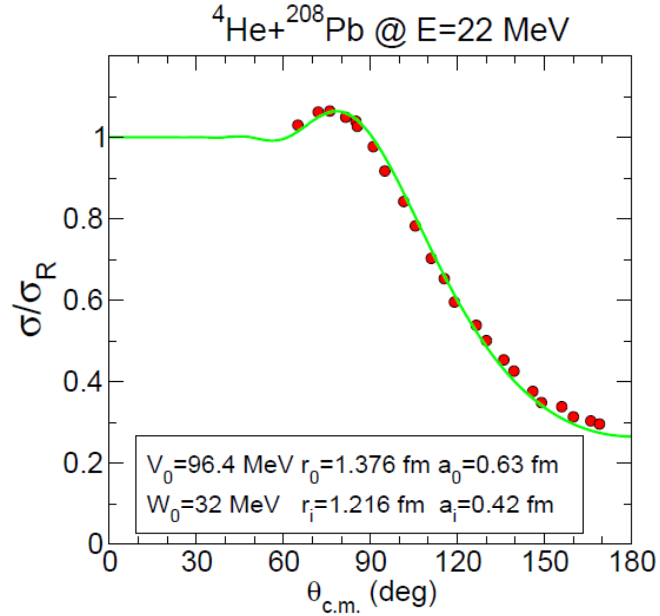


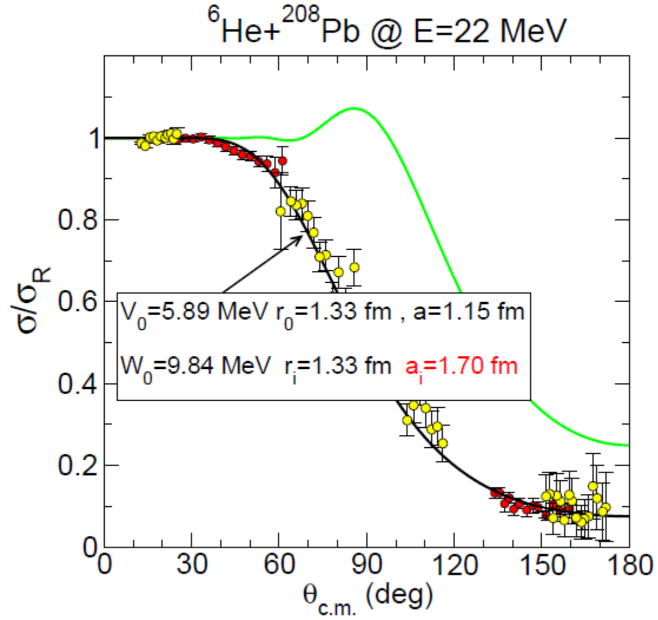
- Bombarding energy around or near the Coulomb barrier
- Coulomb strong ($\eta \gg 1$)
- 'Illuminated' region $\Rightarrow$ interference pattern (near-side/far-side)

  'Shadow' region $\Rightarrow$ strong absorption

elastic scattering: examples

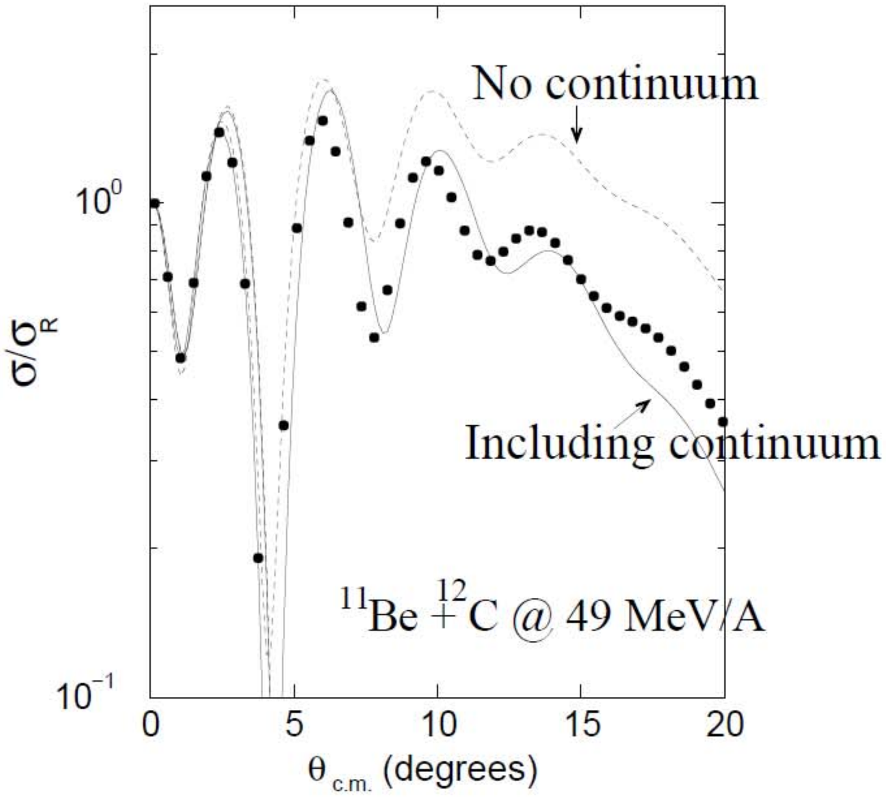
How does the halo structure affect the elastic scatterig?





- ^{4}He+^{208}Pb shows typical Fresnel pattern → *strong absorption*
- ^{6}He+^{208}Pb shows a prominent reduction in the elastic cross section due to the flux going to other channels (mainly break-up)
- ^{6}He+^{208}Pb requires a large imaginary diffuseness → *long-range absorption*

elastic scattering: examples



☞ In Fraunhofer scattering the presence of the continuum produces a reduction of the elastic cross section

reaction and absorptive cross section

Reaction cross section=relates to flux leaving the elastic channel
For simple spherical potentials (single channel) the reaction cross section corresponds to the absorptive cross section

$$\sigma_A = \frac{2}{\hbar v}\frac{4\pi}{k^2}\sum_L (2L+1)\int_0^\infty [-W(R)]\,|\chi_L(R)|^2\,\mathrm{d}R$$

It can be defined more generally in terms of the S-matrix

$$\sigma_R = \frac{\pi}{k^2}\sum_L (2L+1)(1-|\mathbf{S}_L|^2)$$