Training in Advanced Low Energy Nuclear Theory

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Introduction

1 Models for inelastic scattering
- Cluster models: bound states
- Coupling to breakup: CDCC method
- Example for cluster model: ^{11}Be+ ^{12}C

Remainder of Coupled-Channels method: model WF

We expand the total wave function in internal states of the target:

$$\Psi_{\text{model}}(\mathbf{R},\xi) = \phi_0(\xi)\chi_0(\mathbf{R}) + \sum_{n>0} \phi_n(\xi)\chi_n(\mathbf{R})$$

- $\phi_0(\xi)$: target ground state wave function.
- $\chi_0(\mathbf{R})$: describes the projectile-target relative motion with the target in the initial state $\Rightarrow$ elastic scattering

- $\phi_n(\xi)$: target internal wave function in the excited state n.
- $\chi_n(\mathbf{R})$: describes the projectile-target relative motion with the target in the excited state $n \Rightarrow$ inelastic scattering

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- $\chi_n(\mathbf{R})$: describes the projectile-target relative motion with the target in the excited state n ⇒ inelastic scattering

$$\chi_n(\mathbf{R}) \rightarrow \text{unknowns}$$

Calculation of $\chi_n^{(+)}(\mathbf{R})$: the coupled equations

- The model wavefunction must satisfy the Schrödinger equation:

$$[H - E]\Psi^{(+)}_{\rm model}(\mathbf{R}, \xi) = 0$$

- Projecting onto the internal states one gets a system of coupled-equations for the functions $\{\chi_n(\mathbf{R})\}$:

$$[E - \varepsilon_n - T_R - V_{n,n}(\mathbf{R})]\,\chi_n(\mathbf{R}) = \sum_{n' \neq n} V_{n,n'}(\mathbf{R})\chi_{n'}(\mathbf{R})$$

- Coupling potentials:

$$V_{n,n'}(\mathbf{R}) = \int d\xi \phi^*_{n'}(\xi) V(\mathbf{R}, \xi)\phi_n(\xi)$$

☞ *$\phi_n(\xi)$ will depend on the structure model (collective, single-particle,etc).*

Expansion in J, M basis: radial coupled equations

- Expansion of total WF in channel basis:

$$\Psi^{(+)}_{\rm model}(\mathbf{R},\xi) = \sum_{\beta,J,M_J} C^{\beta,JM_J} \frac{\chi^J_\beta(R)}{R} \Phi^{JM_J}_\beta(\hat{R},\xi); \qquad \beta \equiv \{n, L, I\}$$

- The coupled equations:

$$\left(-\frac{\hbar^2}{2\mu}\frac{d^2}{dR^2} + \frac{\hbar^2 L(L+1)}{2\mu R^2} + \epsilon_n - E\right)\chi^J_\beta(R) + \sum_{\beta'} V^J_{\beta,\beta'}(R)\chi^J_{\beta'}(R) = 0$$

- Coupling potentials in the channel basis:

$$V^J_{\beta,\beta'}(R) = \int d\hat{R}\, d\xi \Phi^{JM_J}_\beta(\hat{R},\xi)^* V(\mathbf{R},\xi)\Phi^{JM_J}_{\beta'}(\hat{R},\xi)$$

Solution of the coupled equations: boundary conditions

- For each $J \Rightarrow N$ coupled diff. equations (one for each β) $\Rightarrow N$ indep. solutions
- Standard choice: build solutions characterized by a given entrance channel $\beta_i = \{n_i, L_i, I_i\}$:

$$\Psi^{(+)}_{\beta_i,JM_T}(\mathbf{R},\xi) = \sum_{\beta} \frac{\chi^J_{\beta,\beta_i}(R)}{R} \Phi^{JM_T}_{\beta}(\hat{R},\xi)$$

- The radial functions $\chi^J_{\beta;\beta_i}(R)$ verify:
  - Regular at the origin: $\chi^J_{\beta;\beta_i}(R=0) = 0$
  - Asymptotic behaviour:

$$\chi^J_{\beta;\beta_i}(K_\beta, R) \to \frac{i}{2}\left[H^{(-)}_L(K_\beta R)\delta_{\beta,\beta_i} - S^J_{\beta,\beta_i} H^{(+)}_L(K_\beta R)\right]$$
$$\to \left[F_L(K_\beta R)\delta_{\beta,\beta_i} + T^J_{\beta,\beta_i} H^{(+)}_L(K_\beta R)\right]$$

$$\boxed{S^J_{\beta,\beta_i} = \delta_{\beta,\beta_i} + 2iT^J_{\beta,\beta_i}}$$

(Multi-channel S/T-matrix)

Some properties of the multi-channel S-matrix

- Unitarity of the S-matrix (real potentials):

$$\sum_{\beta} S^{J}_{\beta,\beta_i} S^{J*}_{\beta,\beta_j} = \delta(\beta_i, \beta_f) = \delta(n_i, n_j)\delta(I_i, I_j)\delta(L_i, L_j)$$

- Time reversal: $S^{J}_{nIL,n_iI_iL_i} = S^{J}_{n_iI_iL_i,nIL}$
- Parity conservation: $(-1)^{L}\pi(n) = (-1)^{L_i}\pi(n_i)$.

Strategy

1. Identify the projectile-target interaction $V(\mathbf{R},\xi)$ consistent with our structure model.
2. Perform a multipole expansion:

$$V(\mathbf{R},\xi) = \sqrt{4\pi}\sum_{\lambda,\mu} V_{\lambda\mu}(R,\xi) Y_{\lambda\mu}(\hat{R})$$

3. Evaluate the matrix elements in the basis of internal states:

$$\langle I_f M_f | V(\mathbf{R},\xi) | I_i M_i \rangle = \sqrt{4\pi}\sum_{\lambda,\mu} \frac{\langle I_f M_f | \lambda\mu I_i M_i\rangle}{\sqrt{2I_f+1}} \langle I_f \| V_\lambda(R,\xi) \| I_i \rangle Y_{\lambda\mu}(\hat{R})$$

4. Identify the multipole transition potentials

$$V^{\lambda}_{fi}(R) \equiv \langle I_f \| V_\lambda(R,\xi) \| I_i \rangle = \mathcal{F}_\lambda(R) \langle I_f \| \mathcal{T}^*_\lambda(\xi) \| I_i$$

Single-particle and cluster excitations

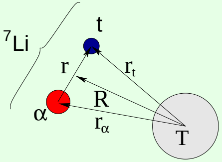
Inelastic scattering: cluster model

- Some nuclei permit a description in terms of two or more clusters: d=p+n, ^{6}Li=α+d, ^{7}Li=α+^{3}H.
- Projectile-target interaction:

$$V(\mathbf{R},\mathbf{r}) = U_1(\mathbf{r}_1) + U_2(\mathbf{r}_2)$$

Example: ^{7}Li=α+t

$$\mathbf{r}_\alpha = \mathbf{R} - \frac{m_t}{m_\alpha + m_t}\mathbf{r}; \quad \mathbf{r}_t = \mathbf{R} + \frac{m_\alpha}{m_\alpha + m_t}\mathbf{r}$$



Internal states:

$$[T_\mathbf{r} + V_{\alpha-t}(\mathbf{r}) - \varepsilon_n]\phi_n(\mathbf{r}) = 0$$

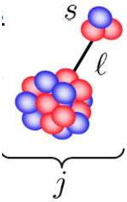
- Transition potentials:

$$V_{n,n'}(\mathbf{R}) = \int d\mathbf{r}\phi_n^*(\mathbf{r})\left[U_1(\mathbf{r}_1) + U_2(\mathbf{r}_2)\right]\phi_{n'}(\mathbf{r})$$

Coupling potentials in the angular momentum basis

- Projectile states:

$$\phi^m_{n\ell j}(\mathbf{r}) = \frac{u_{n\ell j}(r)}{r}[Y_\ell(\hat{r}) \otimes \chi_s]_{jm}$$



(from J.A.Tostevin)

- So, we need to evaluate:

$$\langle (\ell_f s) j_f m_f | V(\mathbf{R}, \xi) | (\ell_i s) j_i m_i \rangle = \langle \phi^{m_f}_{n_f \ell_f j_f} | V(\mathbf{R}, \xi) | \phi^{m_i}_{n_i \ell_i j_i} \rangle$$

Coupling potentials in the angular momentum basis (cont.)

- Multipole expansion of the projectile-target potential

$$V_{vt}(\mathbf{r}_{vt}) + V_{ct}(\mathbf{r}_{ct}) = \sum_{\lambda} (2\lambda + 1)\mathcal{F}_{\lambda}(R, r) \sum Y^{*}_{\lambda\mu}(\hat{r}) Y_{\lambda\mu}(\hat{R})$$

$$\mathcal{F}_{\lambda}(R, r) = \frac{1}{2} \int_{-1}^{1} [V_{vt}(r_{vt}) + V_{ct}(r_{ct})]\, P_{\lambda}(z) d(z); \quad z \equiv \cos(\theta) = \hat{r} \cdot \hat{R}$$

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- General multipole expansion:

$$V(\mathbf{R},\xi) = \sqrt{4\pi}\sum_{\lambda,\mu} V_{\lambda\mu}(R,\xi)Y_{\lambda\mu}(\hat{R}) \Rightarrow \boxed{V_{\lambda\mu}(R,\xi) = \frac{2\lambda+1}{\sqrt{4\pi}}\mathcal{F}_{\lambda}(R,r)Y^{*}_{\lambda\mu}(\hat{r})}$$

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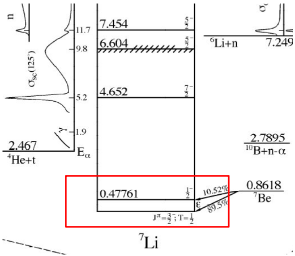
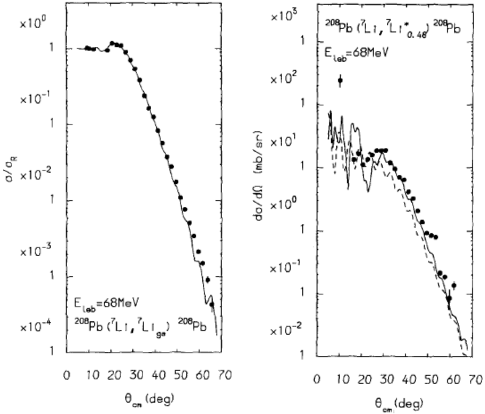
- Multipole coupling potentials

$$\boxed{V^\lambda_{fi}(R) = \langle j_f\|V_\lambda(R,\xi)\|j_i\rangle = \hat{\lambda}\hat{\ell}_i\langle\ell_i 0\ \lambda 0|\ell_f 0\rangle\int_0^\infty u_{n_f\ell_f j_f}(r)\mathcal{F}_\lambda(R,r)u_{n_i\ell_i j_i}(r)dr}$$

Inelastic scattering: cluster model

Example: ^{7}Li(α+t) +^{208}Pb at 68 MeV (Phys. Lett. 139B (1984) 150):
➩ CC calculation with 2 channels (3/2$^-$, 1/2$^-$)

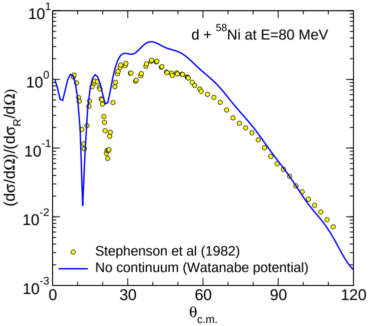


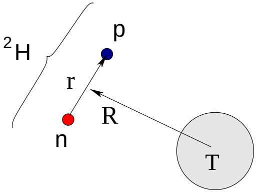


Application of the CC method to weakly-bound systems

Example: Three-body calculation (p+n+^{58}Ni) with Watanabe potential:

$$V_{dt}(\mathbf{R}) = \int d\mathbf{r}\phi_{\rm gs}(\mathbf{r})\left(V_{pt}(\mathbf{r}_{pt}) + V_{nt}(\mathbf{r}_{nt})\right)\phi_{\rm gs}(\mathbf{r})$$





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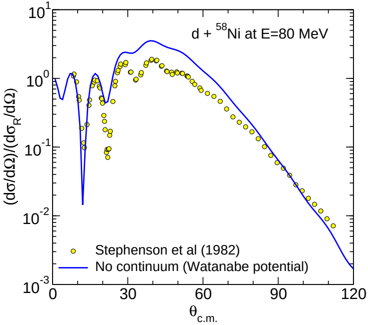
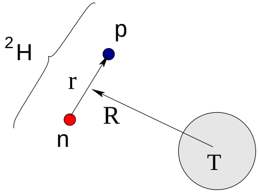
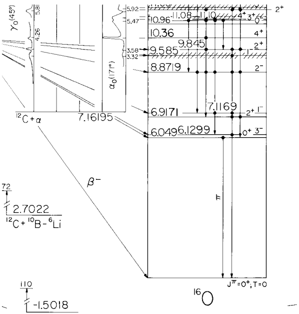


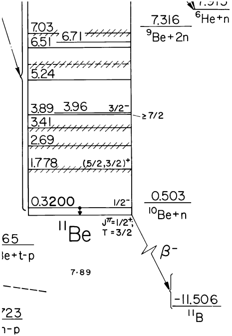


☞*Three-body calculations omitting breakup channels fail to describe the experimental data.*

Inelastic scattering of weakly bound nuclei

- Single-particle (or cluster) excitations become dominant.
- Excitation to continuum states important.

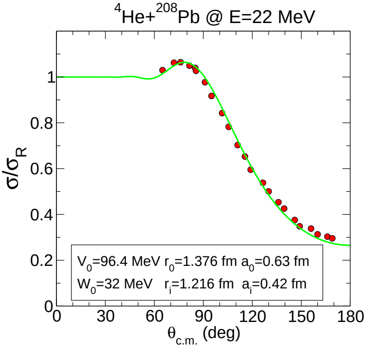


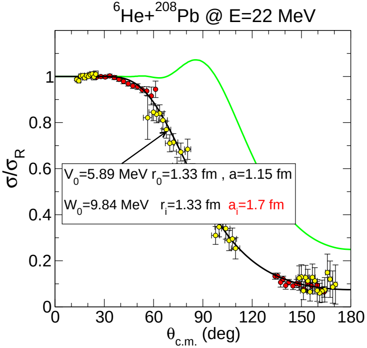


☞*Exotic nuclei are weakly bound ⇒ coupling to continuum states becomes an important reaction channel*

Normal versus halo nuclei

How does the halo structure affect the elastic scattering?





- ^{4}He+^{208}Pb shows typical Fresnel pattern → *strong absorption*
- ^{6}He+^{208}Pb shows a prominent reduction in the elastic cross section due to the flux going to other channels (mainly break-up)

Inclusion of the continuum in CC calculations: continuum discretization

Quantum Hamiltonian

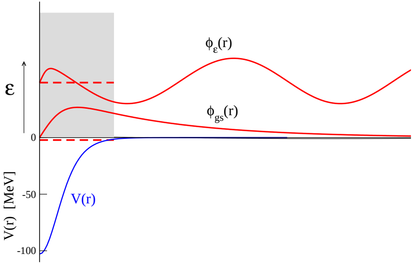
Bound states
- Discrete
- Finite
- Normalizable

Unbound states
- Continuous
- Infinite
- Non-normalizable

Continuum discretization: represent the continuum by a finite set of square-integrable states

***True* continuum**		**Discretized continuum**
Non normalizable	$\rightarrow$	Normalizable
Continuous	$\rightarrow$	Discrete

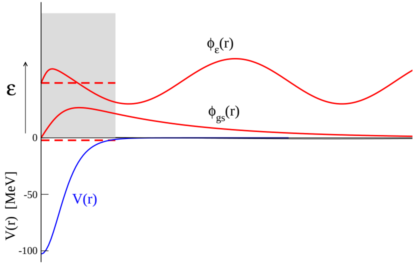
Bound versus scattering states



Continuum state:

$$\phi^{m}_{k,\ell j}(\mathbf{r}) = \frac{u_{k,\ell j}(r)}{r}[Y_{\ell}(\hat{r}) \otimes \chi_s]_{jm}$$

Bound versus scattering states



Continuum state:

$$\phi^m_{k,\ell j}(\mathbf{r}) = \frac{u_{k,\ell j}(r)}{r}[Y_\ell(\hat{r}) \otimes \chi_s]_{jm}$$

Unbound states are not suitable for CC calculations:

- Continuous (infinite) distribution in energy.
- Non-normalizable: $\langle u_{k,\ell sj}(r)^* | u_{k',\ell sj}(r) \rangle \propto \delta(k - k')$

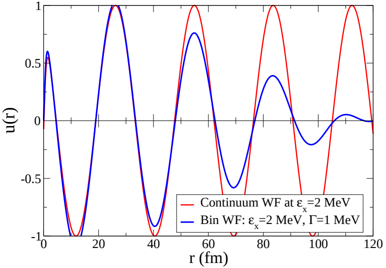
SOLUTION $\Rightarrow$ continuum discretization

CDCC formalism: construction of the bin wavefunctions

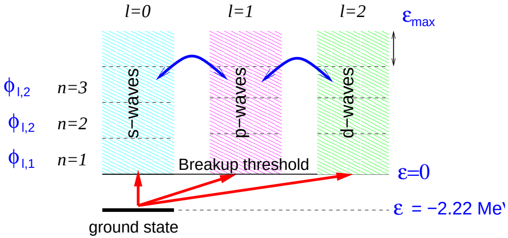
Bin wavefunction:

$$u_{\ell sj,n}(r) = \sqrt{\frac{2}{\pi N}} \int_{k_1}^{k_2} w(k) u_{k,\ell sj}(r) dk$$

- k: linear momentum
- $u_{k,\ell sj}(r)$: scattering states (radial part)
- $w(k)$: weight function



Continuum discretization for deuteron scattering

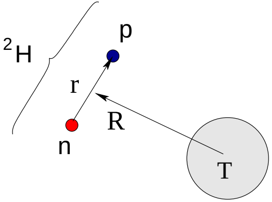


- Select a number of partial waves ($\ell = 0, \ldots, \ell_{max}$).
- For each ℓ, set a maximum excitation energy ε_{max}.
- Divide the interval $\varepsilon = 0 - \varepsilon_{max}$ in a set of sub-intervals (*bins*).
- For each *bin*, calculate a representative wavefunction.

CDCC equations for deuteron scattering

- Hamiltonian:
  $H = T_R + h_r(\mathbf{r}) + V_{pt}(\mathbf{r}_{pt}) + V_{nt}(\mathbf{r}_{nt})$
- Model wavefunction:

$$\Psi(\mathbf{R}, \mathbf{r}) = \phi_{gs}(\mathbf{r})\chi_0(\mathbf{R}) + \sum_{n>0}^{N} \phi_n(\mathbf{r})\chi_n(\mathbf{R})$$



- Coupled equations: $[H - E]\Psi(\mathbf{R}, \mathbf{r}) = 0$

$$\boxed{[E - \varepsilon_n - T_R - V_{n,n}(\mathbf{R})]\chi_n(\mathbf{R}) = \sum_{n' \neq n} V_{n,n'}(\mathbf{R})\chi_{n'}(\mathbf{R})}$$

- Transition potentials:

$$V_{n;n'}(\mathbf{R}) = \int d\mathbf{r} \phi_n(\mathbf{r})^* \left[V_{pt}(\mathbf{R} + \frac{\mathbf{r}}{2}) + V_{nt}(\mathbf{R} - \frac{\mathbf{r}}{2}) \right] \phi_{n'}(\mathbf{r})$$

What observables can we study with CDCC

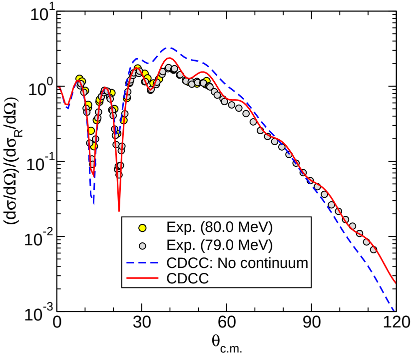
- Elastic scattering
- Breakup angular distribution, as a function of excitation energy:
- Breakup energy distribution, as a function of c.m. angle:

What observables can we study with CDCC

- Elastic scattering
- Breakup angular distribution, as a function of excitation energy:
- Breakup energy distribution, as a function of c.m. angle:

☞ From the S-matrices, more complicated breakup observables can be obtained, such as angular/energy distribution of one of the fragments

Application of the CDCC formalism: d+ ^{58}Ni



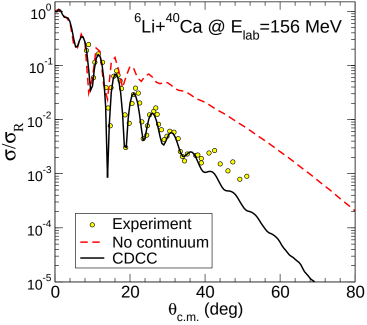
- $\ell = 0, 2$ continuum
- p+^{58}Ni and n+^{58}Ni from Koning-Delaroche OMP.
- $V_{pn}(r) = -72.15\ \exp[-(r/1.484)^2]$

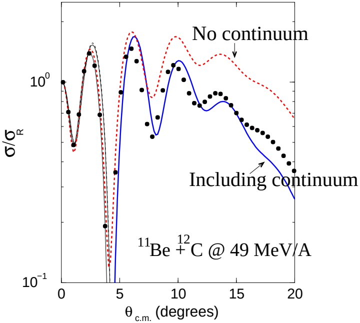
☞ *Coupling to breakup channels has a important effect on the reaction dynamics*

Application of the CDCC method: ^{6}Li and ^{6}He scattering

☞ The CDCC has been also applied to nuclei with a cluster structure:

- ^{6}Li=α + d
- ^{11}Be=^{10}Be + n





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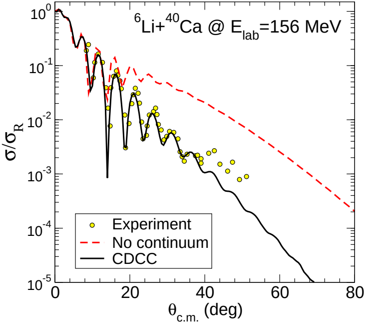
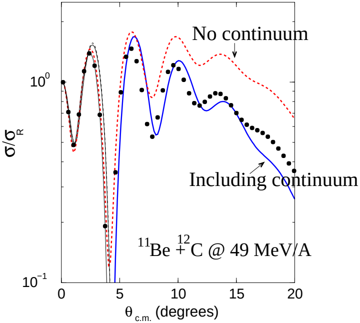
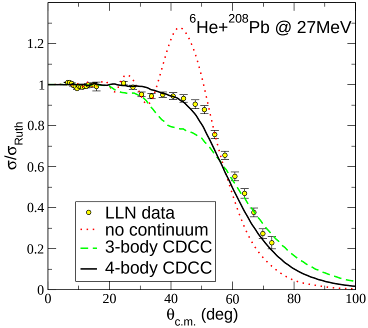


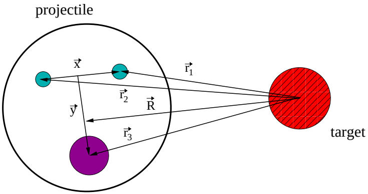


☞ *In Fraunhofer scattering the presence of the continuum produces a reduction of the elastic cross section*

Extension to 3-body projectiles

Eg: ^{6}He=α + n + n





M.Rodríguez-Gallardo et al, PRC 77, 064609 (2008)

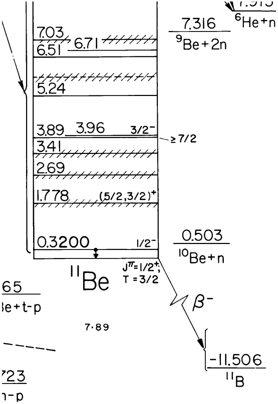
Input example for inelastic scattering within cluster model

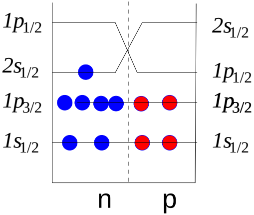
^{11}Be+ ^{12}C $\rightarrow$ ^{11}Be(1/2^{+},1/2^{-}) + ^{12}C at 49.3 MeV/A

Cluster model example: ^{11}Be+^{12}C

Example: ^{11}Be+ ^{12}C $\rightarrow$ ^{11}Be($1/2^{+}$,$1/2^{-}$) + ^{12}C at 49.3 MeV/A

Phys. Rev. C 67, 037601 (2003)





Cluster model example: ^{11}Be+^{12}C

General variables:

```
&FRESCO hcm=0.05 rmatch=60.
     jtmin=0.0 jtmax=150.0      < ---- total angular momentum
     thmin=0.0 thmax=45. thinc=0.5
       iblock=2 smats=2 xstabl=1
       elab=542.3 /
```

- `iblock=2`: number of channels coupled to *all orders*.

Cluster model example: ^{11}Be+^{12}C

Partitions & states:

```
&PARTITION namep='11Be' massp=11. zp=4
          namet='12C' masst=12. zt=6 nex=2 /
&STATES jp=0.5 bandp=1 cpot=1 jt=0.0 bandt=1 /
&STATES jp=0.5 bandp=-1 ep=0.3200 cpot=1 jt=0.0 copyt=1 /
```

- `nex=2`: This partition will contain two pairs of states.
- `copyt=1`: The target of the second pair of states is just the same (a copy) of the first target stat.

```
&PARTITION namep='10Be' massp=10.0 zp=4
          namet='12C+n' masst=13.0 zt=6 nex=1 /
&STATES jp=0.0 bandp=1 cpot=2 jt=0.0 bandt=1 /
```

Cluster model example: ^{11}Be+^{12}C

Projectile-target Coulomb potential (monopole):

```
&POT kp=1 ap=11.0 at=12.0 rc=1.111 /
```

Neutron-target & core-target potentials:

```
&POT kp=3 ap=0.0 at=12.0 rc=1.111 /
&POT kp=3 type=1 p1=37.4 p2=1.2 p3=0.75
              p4=10.0 p5=1.3 p6=0.6 /
```

```
&POT kp=2 ap=10.0 at=12.000 rc=1.111 /
&POT kp=2 type=1 p1=123.0 p2=0.75 p3=0.8
               p4=65.0 p5=0.78 p6=0.8 /
```

Neutron binding potential:

```
&POT kp=4 ap=0 at=10.0 rc=1.0 /
&POT kp=4 type=1 p1=87.0 p2=1.0 p3=0.53 /
```

^{11}Be+^{12}C inelastic scattering

Bound state wave functions (overlaps):

```
&OVERLAP kn1=1 ic1=1 ic2=2 in=1 nn=2 sn=0.5 l=0 j=0.5
       kbpot=4 be=0.500 isc=1 /
&OVERLAP kn1=2 ic1=1 ic2=2 in=1 nn=1 l=1 sn=0.5
       j=0.5 kbpot=4 be=0.180 isc=1 ipc=2 /
```

- `kn1`: Index for this WF
- `ic1/ic2`: Index of partition containing core (^{10}Be) / composite (^{11}Be)
- `in`: WF for projectile (in=1) or target (in=2)
- `nn,sn,l,j`: Quantum numbers for bound state
- `be`: separation energy.
- `kbpot`: Index KP of binding potential.

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Couplings:

```
&COUPLING icto=1 icfrom=2 kind=3 ip1=4 ip2=1 p1=3.0 p2=2.0 /
```

- `kind=3`: Single-particle excitations of projectile
- `icto=1`: Partition containing nucleus being excited (^{11}Be)
- `icfrom=2`: Partition containing core (^{10}Be)
- `ip1=4`: Maximum multipole for coupling potentials
- `p1/p2`: KP index for fragment-target / core-target potentials

Spectroscopic amplitudes:

```
&CFP in=1 ib=1 ia=1 kn=1 a=1.000 /
&CFP in=1 ib=2 ia=1 kn=2 a=1.000 /
```

- `in=1/2`: Projectile/target
- `ib/ia`: Index for composite/core state
- `a`: Spectroscopic amplitude