The R-matrix Method: phenomenological variant

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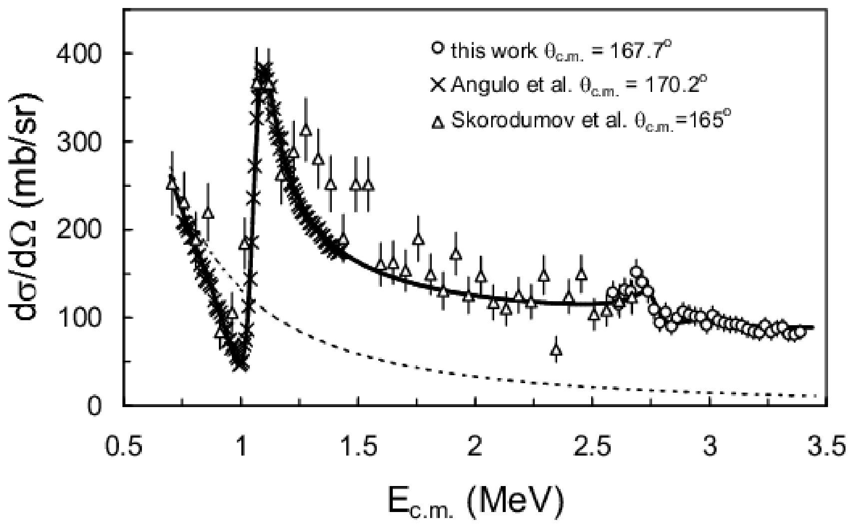
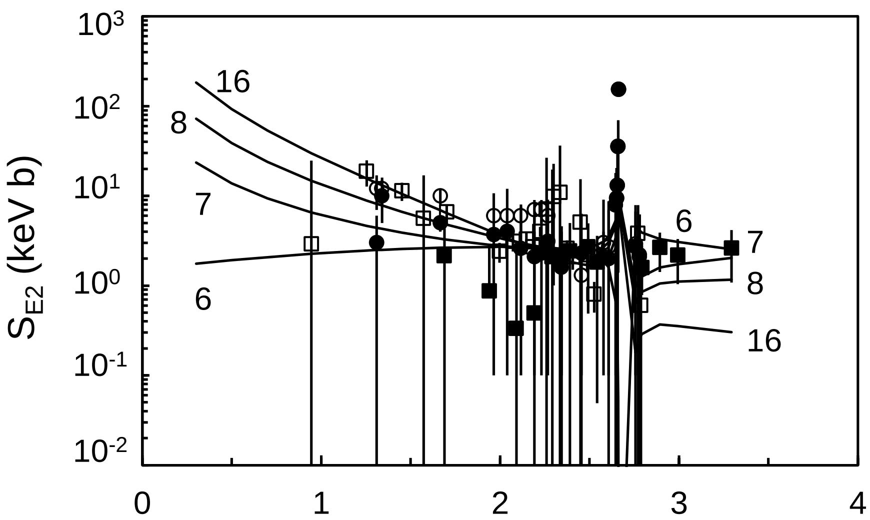
1. Introduction
2. Derivation of the « phenomenological » R matrix
3. Observed vs calculated parameters
4. Applications of the « phenomenological » R matrix
5. Extension to inelastic scattering and transfer
6. Conclusions

1. Introduction

Main Goal: fit of experimental data



^{18}Ne+p elastic scattering
→ resonance properties



Nuclear astrophysics: $^{12}C(\alpha,\gamma)^{16}O$
→Extrapolation to low energies

1. Introduction

Experimental data: cross sections

Theory: cross section $\Leftarrow$ scattering matrices $U_\ell(E) \Leftarrow$ R matrices $R_\ell(E)$

Phenomenological R matrix:

- Definition: $R(E) = \sum_{\lambda=1}^{N} \frac{\gamma_\lambda^2}{E_\lambda - E}$
- Set of parameters $E_\lambda, \gamma_\lambda^2$ used as fitting parameters

Questions:

- Origin of the R-matrix definitions? Link with the calculable R-matrix?
- Link between the R-matrix parameters (« calculated ») and the experimental energies and widths (« observed »)?

2. Derivation of the phenomenological R-matrix

General input for R matrix calculations

- Channel radius a
- Set of N basis functions $\phi_i(r)$

General definition of the R matrix

$$R(E) = \frac{\hbar^2 a}{2\mu} \sum_{ij} \phi_i(a)\,(D^{-1})_{ij}\phi_j(a)$$

With

- Matrix element $D_{ij} =< \phi_i|H - E + \mathcal{L}(L)|\phi_j >_{int}$
- Bloch operator $\mathcal{L}(L) = \frac{\hbar^2}{2\mu a}\delta(r-a)\left(\frac{d}{dr} - \frac{L}{r}\right)r$
- Constant $L = ka\frac{O'(ka)}{O(ka)} = S(E) + iP(E)$ (S=shift factor, P=penetration factor)

Provides the scattering matrix $U(E) = \frac{I(ka)}{O(ka)}\frac{1-L^*R(E)}{1-LR(E)}$

2. Derivation of the phenomenological R-matrix

Defintion of the R matrix $R(E) = \frac{\hbar^2 a}{2\mu} \sum_{ij} \phi_i(a)\, (D^{-1})_{ij} \phi_j(a)$

$D_{ij} =< \phi_i | H - E + \mathcal{L}(L) | \phi_j >_{int}$

Valid for any basis (matrix D must be computed with the same phi)

→ Particular choice: eigenstates of $H + \mathcal{L}(L)$

→ u_λ defined by $\big(H + \mathcal{L}(L)\big) u_\lambda(r) = E_\lambda u_\lambda(r)$

Then $D_{\lambda\lambda'} =< u_\lambda | H - E + \mathcal{L}(L) | u_{\lambda'} >_{int} = (E_\lambda - E)\delta_{\lambda\lambda'}$

$(D^{-1})_{\lambda\lambda'} = \frac{1}{E_\lambda - E} \delta_{\lambda\lambda'}$

The R-matrix can be rewritten as $R(E) = \frac{\hbar^2 a}{2\mu} \sum_\lambda \frac{u_\lambda(a)^2}{E_\lambda - E} = \sum_\lambda \frac{\gamma_\lambda^2}{E_\lambda - E}$

With $\gamma_\lambda^2 = \frac{\hbar^2 a}{2\mu} u_\lambda(a)^2$ = reduced width (always positive)

- γ_λ is real and energy independent
- $\gamma_\lambda \sim$ wave function at the channel radius → measurement of the clustering
- dimensionless reduced width $\theta^2 = \frac{\gamma^2}{\gamma_W^2}$ where $\gamma_W^2 = \frac{3\hbar^2}{2\mu a^2}$ is the Wigner limit ($\theta^2 < 1$)

2. Derivation of the phenomenological R-matrix

Calculation of the eigenstates: expansion over a basis

$$u_\lambda(r) = \sum_i c_i^\lambda \phi_i(r)$$

$$H_{ij} = < \phi_i | H | \phi_j >_{int} = \int_0^a \phi_i(r) H \phi_j(r) dr$$

$$N_{ij} = < \phi_i | \phi_j >_{int} = \int_0^a \phi_i(r) \phi_j(r) dr$$

Determine the N eigenvalues and eigenvectors of

$\sum_{j=1}^{N} c_j^\lambda \left(H_{ij} - E_\lambda N_{ij}\right) = 0$ (simple matrix diagonalization)

with $\lambda =$ poles: depend on a (matrix elements H_{ij} and N_{ij} depend on a)

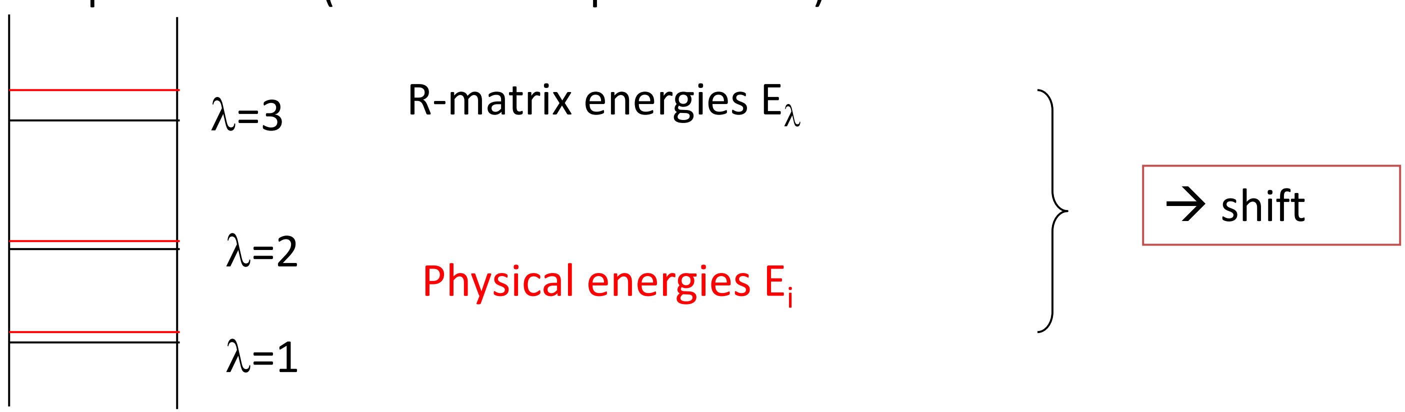
2. Derivation of the phenomenological R-matrix

General definition of the R-matrix: $R(E) = \sum_{\lambda=1}^{N} \frac{\gamma_\lambda^2}{E_\lambda - E}$

- Calculable R-matrix: parameters $E_\lambda, \gamma_\lambda^2$ are **calculated from basis functions**
- Phenomenological R-matrix: parameters are **fitted to data** (phase shifts, cross sections, etc.)
- Must be done for each ℓ value → adapted to low level densities
- In general: single-pole approximation $R(E) \approx \frac{\gamma_0^2}{E_0 - E}$

Main problem: what is the link between

- The R matrix parameters (="formal" parameters)
- The physical parameters (="observed" parameters)

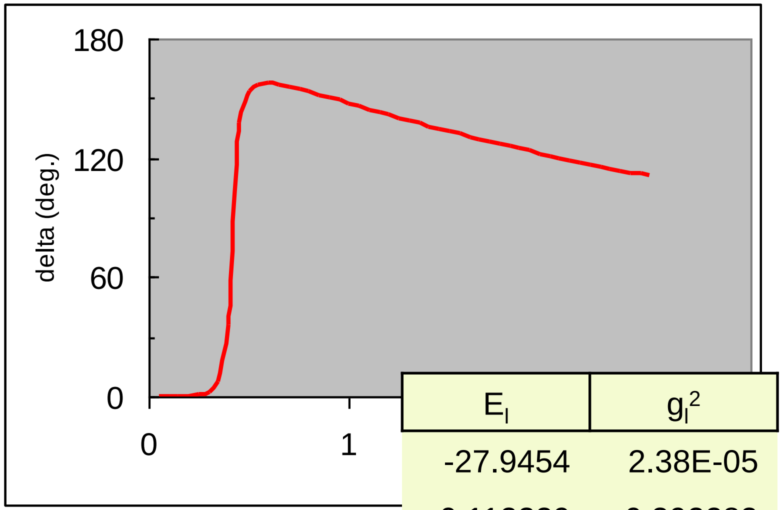


2. Derivation of the phenomenological R-matrix

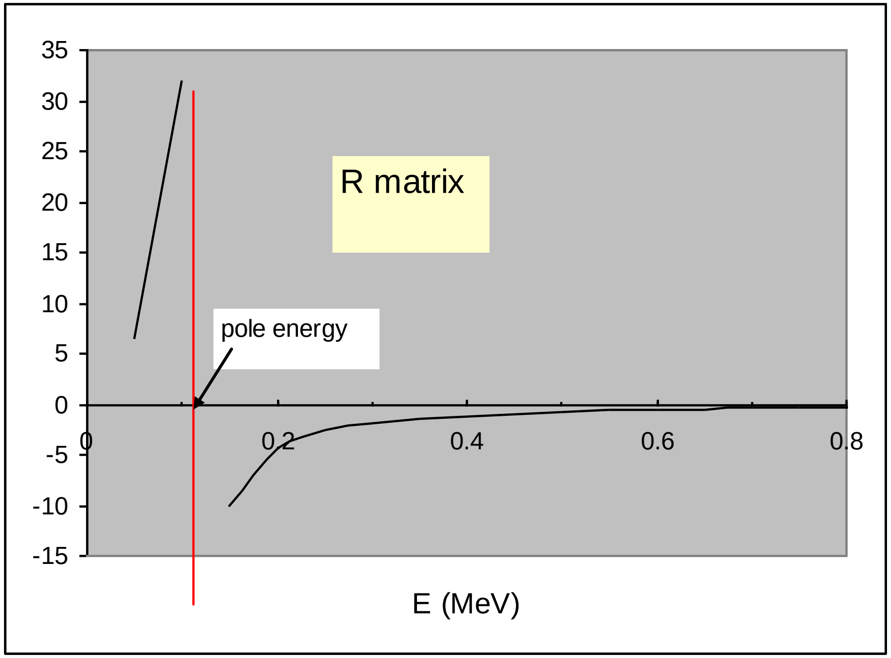
Example : $^{12}C+p$

- potential : $V=-70.5*\exp(-(r/2.70)^2)$
- Basis functions: $u_i(r)=r^{\ell}*\exp(-(r/a_i)^2)$ with $a_i=x_0*a_0^{(i-1)}$

10 basis functions, a=8 fm



$$R(E)=\sum_{\lambda}\frac{\gamma_{\lambda}^{2}}{E_{\lambda}-E}$$



10 eigenvalues

E_l	g_l^2
-27.9454	2.38E-05
0.112339	0.392282
7.873401	1.094913
26.50339	0.730852
40.54732	0.010541
65.62125	1.121593
107.7871	4.157215
153.8294	1.143461
295.231	0.097637
629.6709	0.019883

2. Derivation of the phenomenological R-matrix

Calculation: 10 poles

pole	E_l	γ_l^2
1	-27.95	2.38E-05
2	0.11	3.92E-01
3	7.87	1.09E+00
4	26.50	7.31E-01
5	40.55	1.05E-02
6	65.62	1.12E+00
7	107.79	4.16E+00
8	153.83	1.14E+00
9	295.23	9.76E-02
10	629.67	1.99E-02

Fit to data

Isolated pole (2 parameters)

$$\frac{\gamma_0^2}{E_0 - E}$$

Background (high energy): gathered in 1 term

$$R_0(E) = \sum_{\lambda \neq 0} \frac{\gamma_\lambda^2}{E_\lambda - E}$$

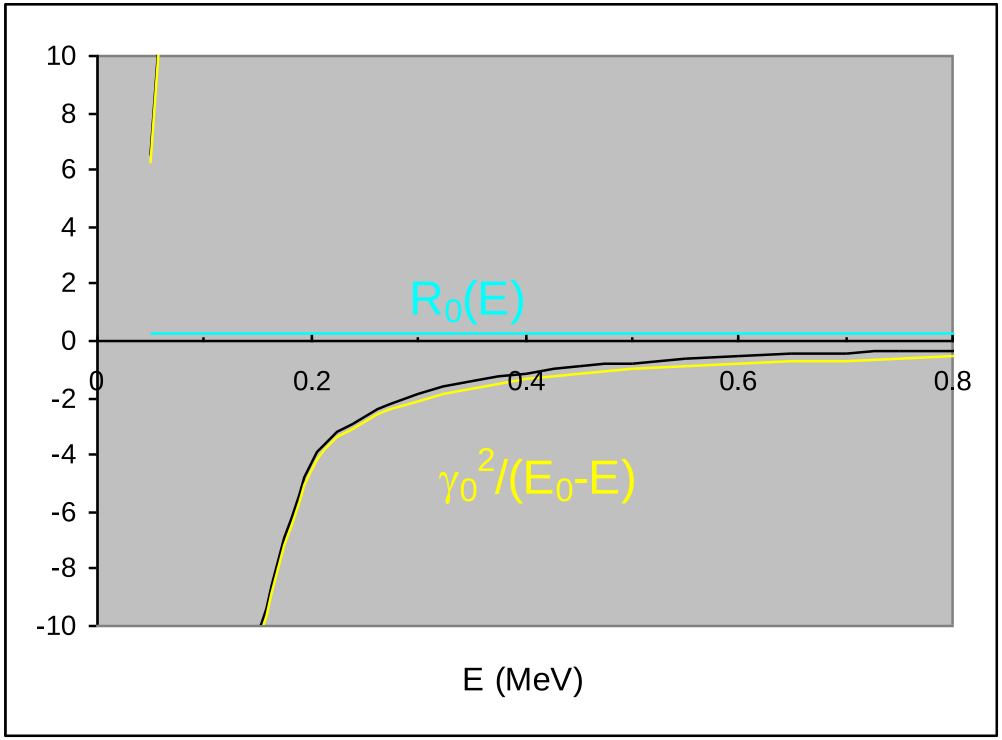
$E << E_l \rightarrow R_0(E) \sim R_0$

→ In phenomenological approaches (one resonance): $R(E) \approx \frac{\gamma_0^2}{E_0 - E} + R_0$

2. Derivation of the phenomenological R-matrix

$^{12}C+p$

$$R(E)=\sum_\lambda \frac{\gamma_\lambda^2}{E_\lambda - E}=\frac{\gamma_0^2}{E_0-E}+R_0(E)$$



Approximations: $R_0(E)=R_0$=constant (background)

$R_0(E)=0$: Breit-Wigner approximation: one term in the R matrix

Remark: the R matrix method is NOT limited to resonances ($R=R_0$)

3. "Observed" vs "calculated" R-matrix parameters

Question: how to determine the resonance properties (energy, width) from the R-matrix data?

Relation between the collision matrix and the R matrix

$$U^\ell = \frac{I_\ell(ka)}{O_\ell(ka)} \frac{1 - L^\star R^\ell}{1 - LR^\ell}, \text{ with } L(E) = S(E) + iP(E)$$
$$= \exp(2i\delta^\ell) = \exp(2i(\delta^\ell_{HS} + \delta^\ell_R))$$

with

$$\exp(2i\delta^\ell_{HS}) = \frac{I_\ell(ka)}{O_\ell(ka)} \rightarrow \delta^\ell_{HS} = -\arctan \frac{F_\ell(ka)}{G_\ell(ka)}$$

Hard-sphere

$$\exp(2i\delta^\ell_R) = \frac{1 - L^\star R^\ell}{1 - LR^\ell} \rightarrow \delta^\ell_R = \arctan \frac{PR}{1 - SR}$$

R-matrix

Resonance energy E_r defined by $1 - S(E_r)R(E_r) = 0$ → $\delta_R = 90^o$

In general: must be solved numerically

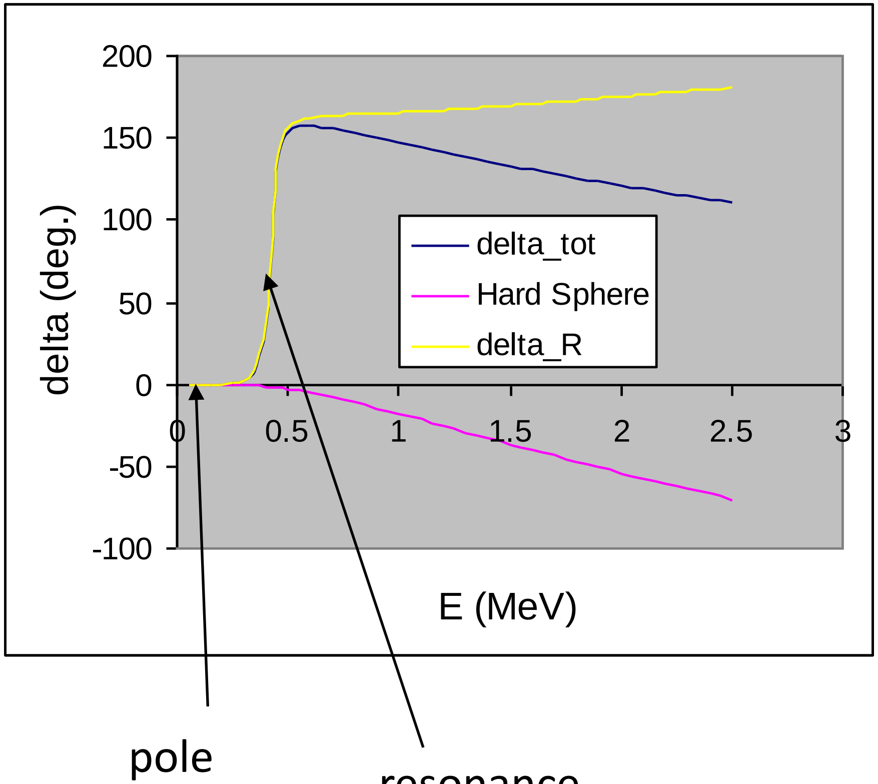
3. “Observed” vs “calculated” R-matrix parameters

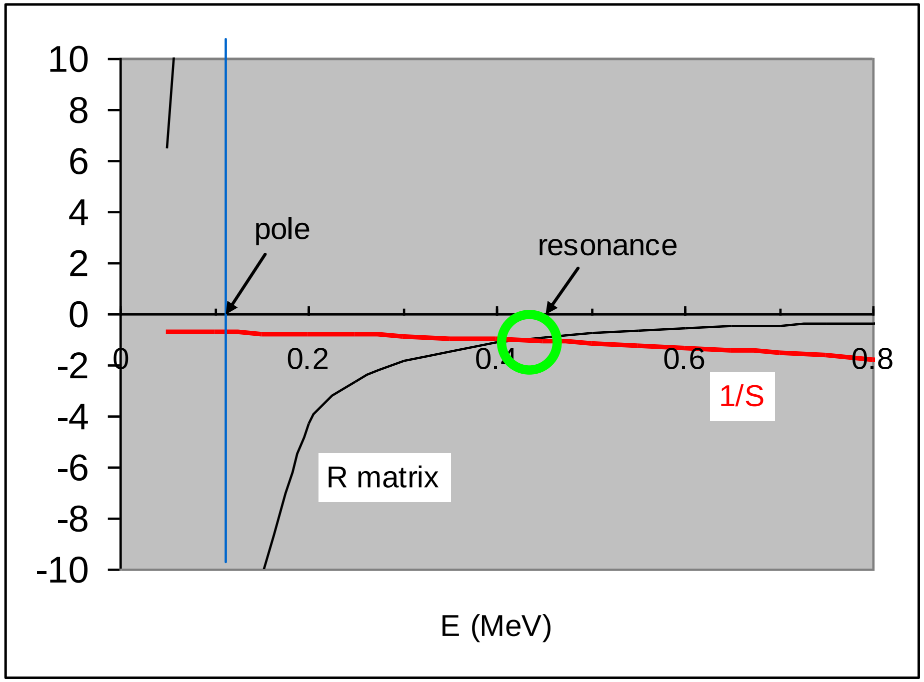
$$\delta_R^\ell(E) = \arctan \frac{P(E)R(E)}{1 - S(E)R(E)}$$

Resonance energy E_r defined by $1 - S(E_r)R(E_r) = 0$ →δ_R=90°

In general: must be solved numerically

Plot of R(E), 1/S(E) for ^{12}C+p



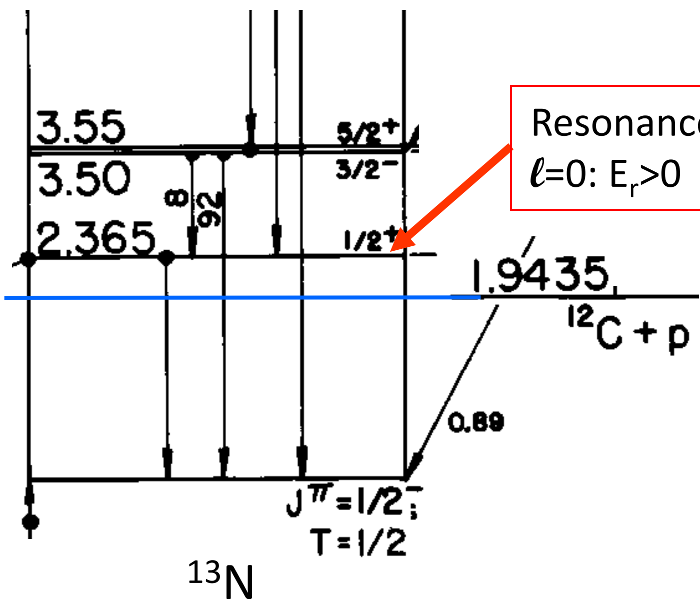


3. “Observed” vs “calculated” R-matrix parameters

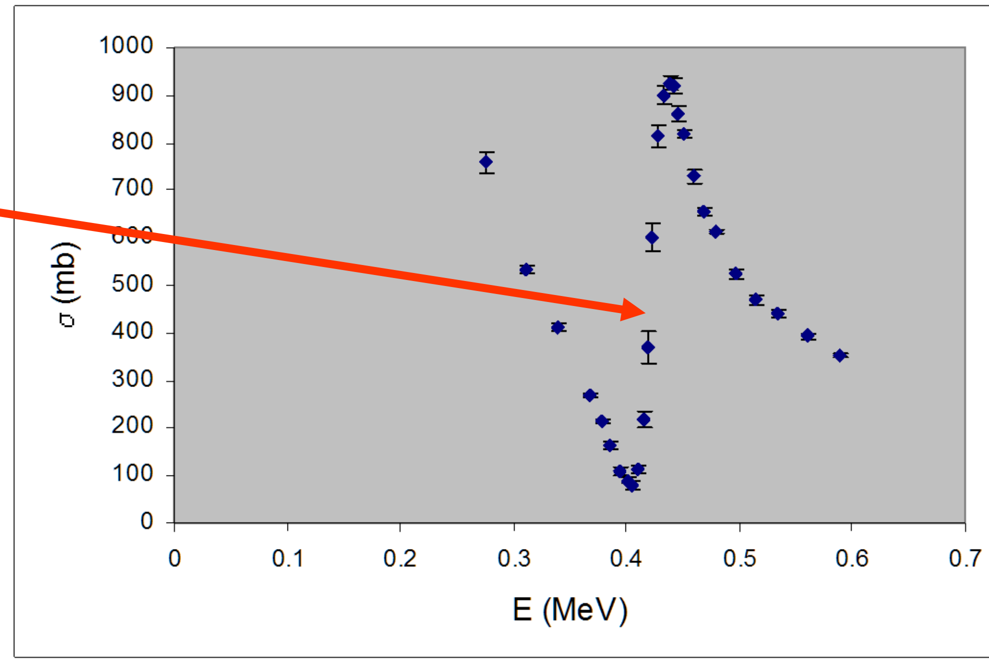
Simple case: one pole (=“isolated” resonance):

$$R(E) = \frac{\gamma_0^2}{E_0 - E}$$

Example: ^{12}C+p: E_R=0.42 MeV



Resonance
ℓ=0: E_r>0



Question: link between E_0 and E_R? ➔ Breit-Wigner

3. "Observed" vs "calculated" R-matrix parameters

The Breit-Wigner approximation

Single pole in the R matrix expansion: $R(E) = \frac{\gamma_0^2}{E_0 - E}$

Phase shift: $\tan \delta_R(E) = \frac{P(E)R(E)}{1 - S(E)R(E)} \approx \frac{\gamma_0^2 P(E)}{E_0 - E - \gamma_0^2 S(E)}$

$$\approx \frac{\Gamma(E)}{2(E_r - E)}$$

Thomas approximation: shift function linear → $S(E) \approx S(E_0) + S'(E_0)(E - E_0)$

Then: $E_r \approx E_0 - \frac{\gamma_0^2 S(E_0)}{1 + \gamma_0^2 S'(E_0)}$

$$\Gamma(E) = 2\frac{\gamma_0^2}{1 + \gamma_0^2 S'(E_r)} P(E) = 2\gamma_{obs}^2 P(E)$$

At the resonance: $\Gamma_r = \Gamma(E_r) = 2\gamma_{obs}^2 P(E_r)$: strongly depends on energy

→ Breit-Wigner = R-matrix with one pole

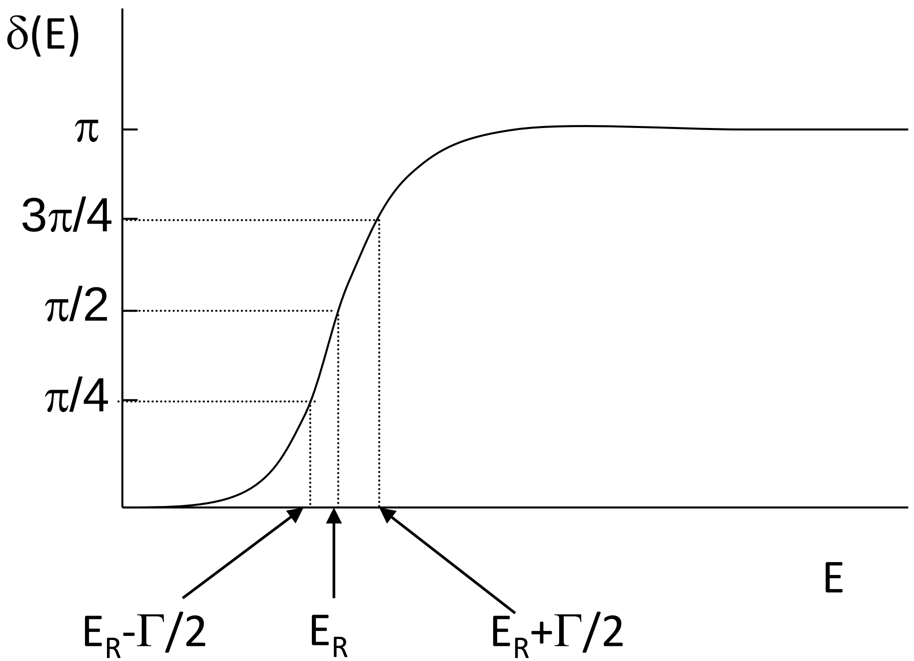
→ Generalization possible (more than one pole: interference effects)

3. "Observed" vs "calculated" R-matrix parameters

Phase shift for one pole

Breit-Wigner approximation: $\tan \delta_R \approx \frac{\Gamma}{2(E_R - E)}$ =one-pole R matrix

E_R=resonance energy
Γ=resonance width



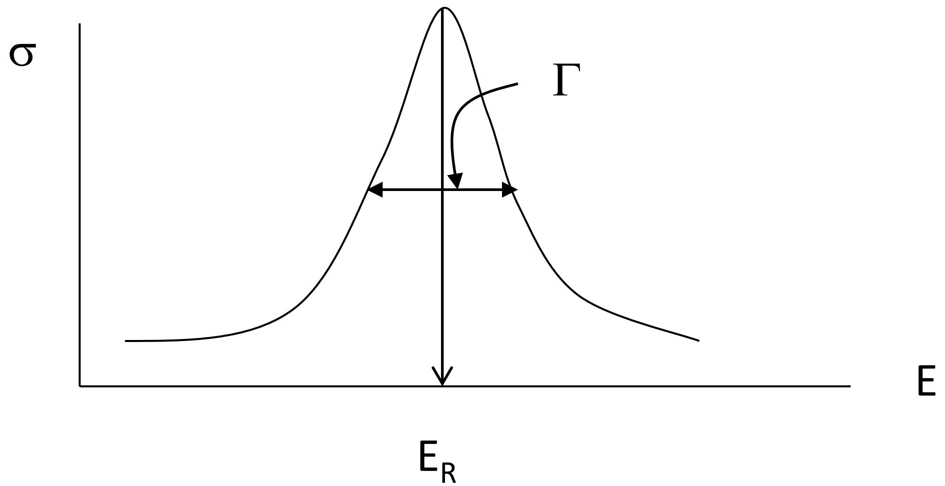
- Narrow resonance: Γ small
- Broad resonance: Γ large

3. "Observed" vs "calculated" R-matrix parameters

Cross section

$\sigma(E) = \frac{\pi}{k^2}\sum_\ell (2\ell+1)|\exp(2i\delta_\ell)-1|^2$ maximum for $\delta = \frac{\pi}{2}$

Near the resonance: $\sigma(E) \approx \frac{4\pi}{k^2}(2\ell_R+1)\frac{\Gamma^2/4}{(E_R-E)^2+\Gamma^2/4}$, where ℓ_R=resonant partial wave



In practice:
- Peak not symmetric (Γ depends on E)
- « Background » neglected (other ℓ values)
- Differences with respect to Breit-Wigner

3. “Observed” vs “calculated” R-matrix parameters

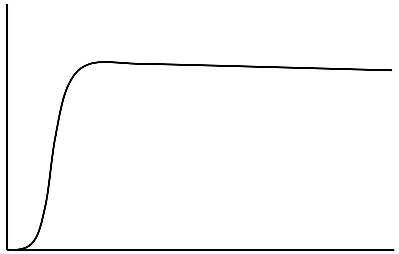
Narrow vs broad resonances

Comparison of 2 caracteristic times
a. Resonance lifetime $\tau_R = \hbar/\Gamma$
b. Interaction time (no resonance): $\tau_{NR} = d/v$

Example1: ^{12}C+p

Resonance properties: E_R=0.42 MeV, Γ=32 keV → lifetime: τ_R = ~2x10^{-20} s
Interaction range d~10 fm →interaction time τ_{NR}~1.1x10^{-21} s

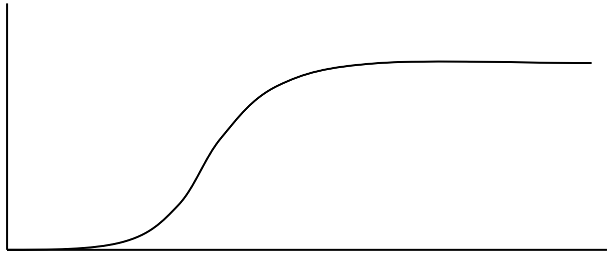
→ narrow resonance



Example 2: α+p

Resonance properties: E_R=1.72 MeV, Γ=1.2 MeV → lifetime: τ_R = ~6x10^{-22} s
Interaction range d~10 fm →interaction time τ_{NR}~5x10^{-22} s

→ broad resonance



3. "Observed" vs "calculated" R-matrix parameters

Comments on the Breit-Wigner approximation

$$\tan \delta_R(E) \approx \frac{\Gamma(E)}{2(E_R - E)}$$

- One-pole approximation, neglects background effects
- From a one-pole R matrix:

$$\tan \delta_R(E) = \frac{\gamma_0^2 \, P(E,a)}{E_0 - E - \gamma_0^2 S(E,a)}$$

→ Depends on the channel radius a

- Equivalent to

$\tan \delta_R(E) = \frac{\gamma_0^2 \, P(E,a)}{E_R - E - \gamma_0^2 (S(E,a) - S(E_R,a))}$ additional term in the denominator

Using $S(E) \approx S(E_R) + (E - E_R) S'(E_R)$

Equivalent to

$\tan \delta_R(E) = \frac{\gamma_{obs}^2 \, P(E,a)}{E_R - E} = \frac{\Gamma(E)}{2(E_R - E)}$, with $\gamma_{obs}^2 = \frac{\gamma_0^2}{1 + \gamma_0^2 S'(E_R)}$

If the energy dependence of $\Gamma(E)$ is neglected → $\tan \delta_R(E) \approx \frac{\Gamma(E_R)}{2(E_R - E)}$

➔ Several possible definitions of the Breit-Wigner approximation

➔ The parameters depend (to some extend) on the definition

3. "Observed" vs "calculated" R-matrix parameters

Link between "calculated" and "observed" parameters

One pole (N=1)

$$E_R = E_0 - \frac{S(E_0)\gamma_0^2}{1 + S'(E_0)\gamma_0^2}$$

$$\gamma_{obs}^2 = \frac{\gamma_0^2}{1 + S'(E_0)\gamma_0^2}$$

R-matrix parameters (calculated)

↕

Observed parameters (=data)

Several poles (N>1)

$$1 - S(E_r)R(E_r) = 0$$

Must be solved numerically

Generalization of the Breit-Wigner formalism:
link between observed and formal parameters when *N*>1

C. Angulo, P.D., Phys. Rev. C **61**, 064611 (2000) – single channel
C. Brune, Phys. Rev. C **66**, 044611 (2002) – multi channel

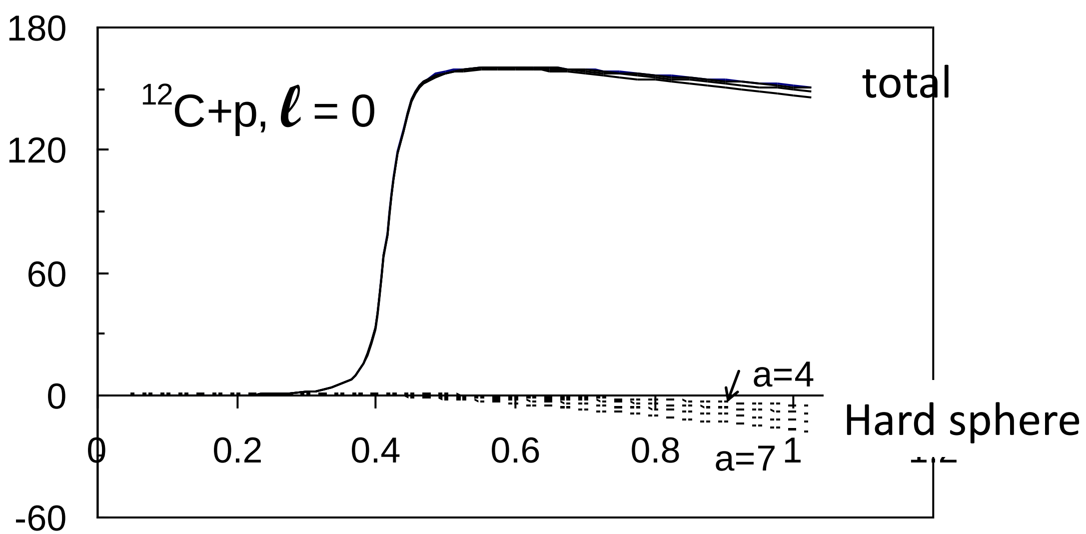
3. “Observed” vs “calculated” R-matrix parameters

Examples: ^{12}C+p and $^{12}C+\alpha$

Narrow resonance: ^{12}C+p

^{12}C+p ($E^r = 0.42$ MeV, $\Gamma = 32$ keV, $J = 1/2^+, \ell = 0$)

	$a = 4$ fm	$a = 5$ fm	$a = 6$ fm	$a = 7$ fm
γ^2_{obs} (MeV)	1.09	0.59	0.35	0.23
E_0 (MeV)	−2.15	−0.61	−0.11	0.11
γ^2_0 (MeV)	3.09	1.16	0.57	0.32

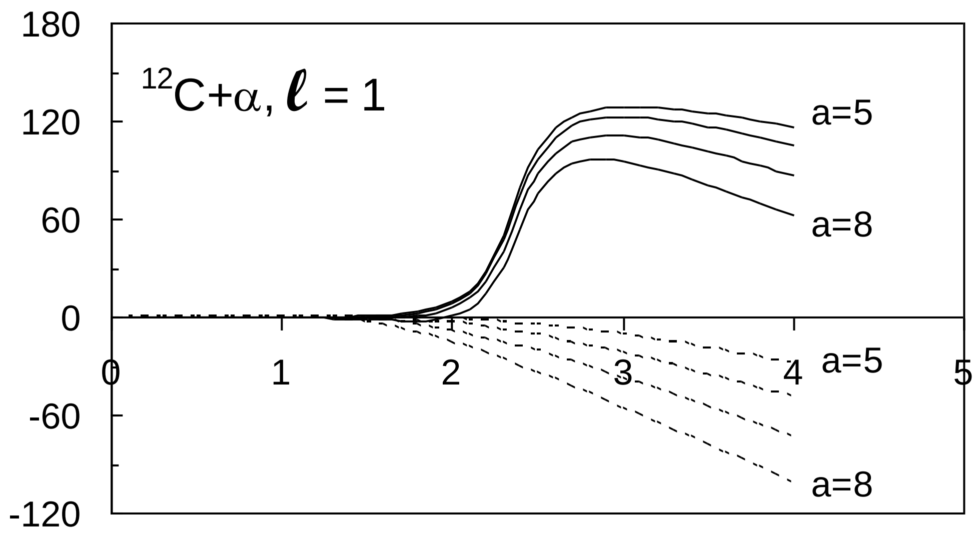


3. “Observed” vs “calculated” R-matrix parameters

Broad resonance: $^{12}C+\alpha$

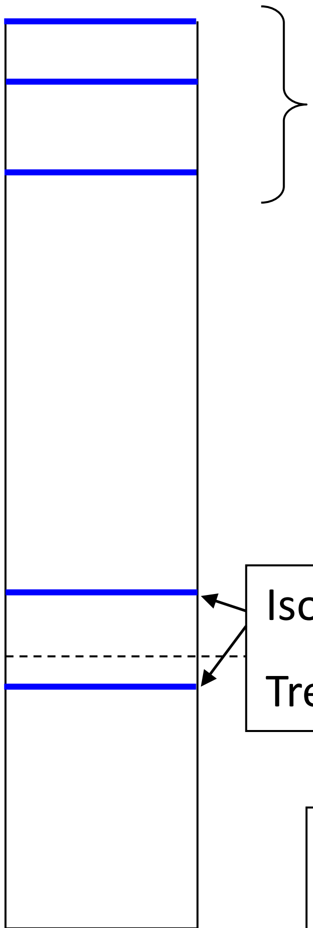
$^{12}C+\alpha$ ($E^r = 2.42$ MeV, $\Gamma = 0.42$ MeV, $J = 1^-, \ell = 1$)

	$a = 5$ fm	$a = 6$ fm	$a = 7$ fm
γ^2_{obs} (MeV)	0.57	0.28	0.16
E_0 (MeV)	0.49	1.92	2.22
γ^2_0 (MeV)	1.17	0.37	0.19



3. “Observed” vs “calculated” R-matrix parameters

Extension to multi-resonances



High-energy states with the same J^{π}

Simulated by a single pole = **background (R_0)**

Isolated resonances:

Treated individually

Energies of interest: E

In this case: $$R(E) = \frac{\gamma_1^2}{E_1 - E} + \frac{\gamma_2^2}{E_2 - E} + R_0$$

Non resonant calculations possible: $R(E) = R_0$

4. Applications to elastic scattering

Cross section

$\frac{d\sigma}{d\Omega} = |f(\theta)|^2$, with $f(\theta) = \frac{1}{k}\sum_\ell (1 - \exp(2i\delta_\ell))\,(2\ell + 1) P_\ell(\cos\theta)$

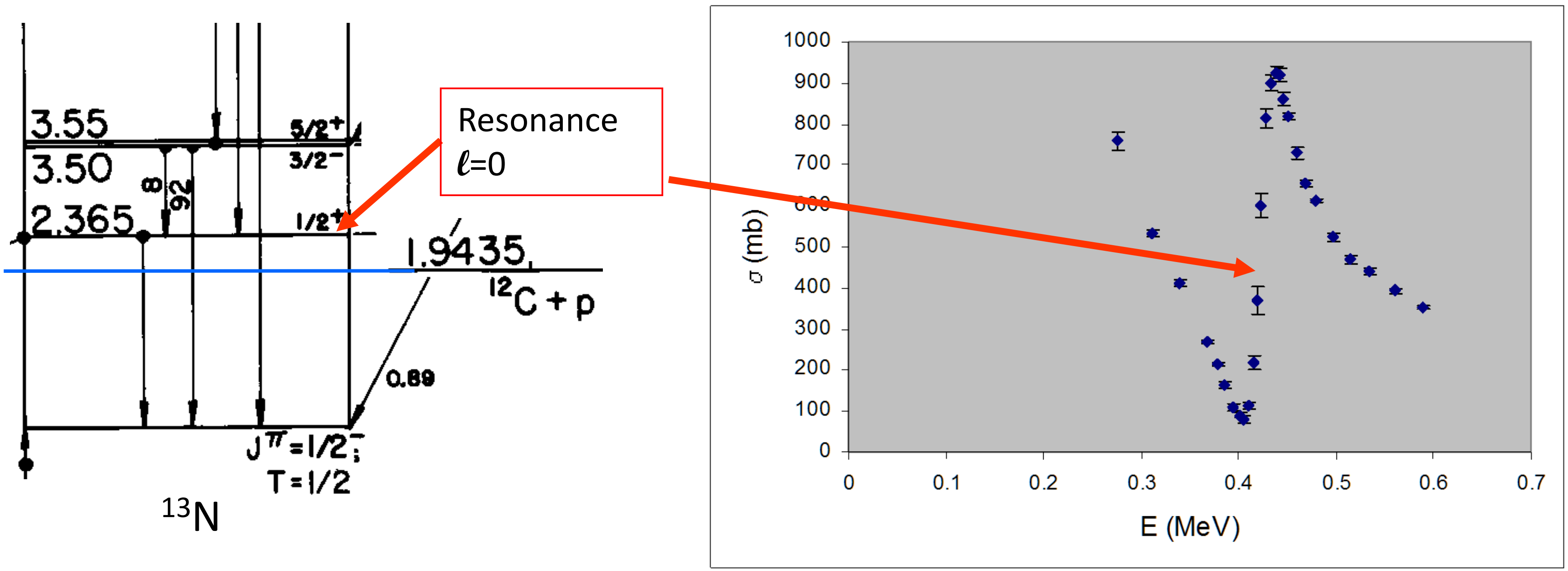
→ How to obtain the phase shifts δ_ℓ

General strategy: identify which resonances are important

- **Resonant** partial waves: R-matrix (parameters fitted or obtained from external data)
- **Non-resonant** patial waves
  - $\delta_\ell = 0$
  - Or δ_ℓ= hard-sphere (consistent with R-matrix R=0)

4. Applications to elastic scattering

Example: $^{12}C+p$: $E_R=0.42$ MeV



In the considered energy range: resonance J=1/2+ (l=0)

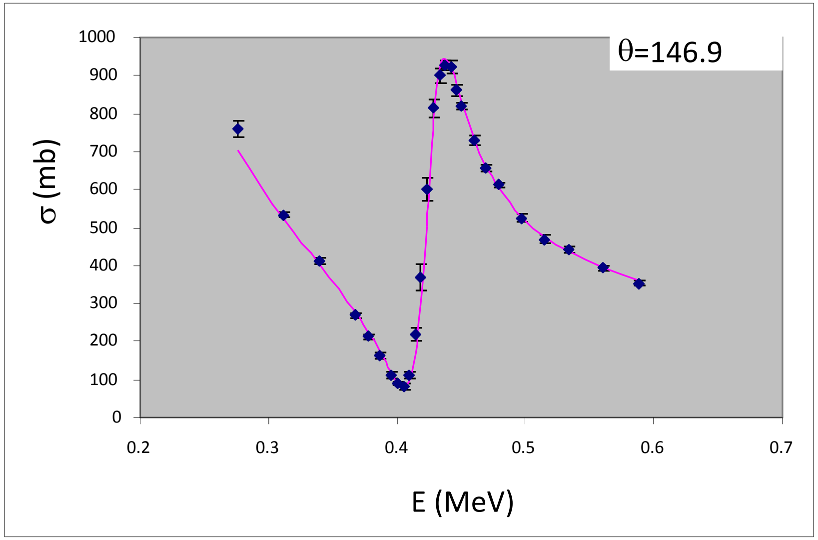
→ Phase shift for l=0 is treated by the R matrix

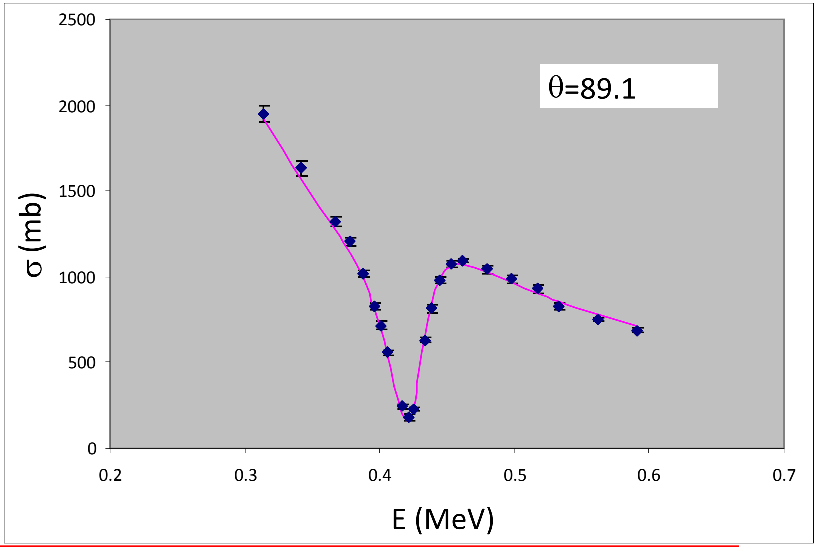
→ Other phase shifts are given by the hard-sphere approximation

4. Applications to elastic scattering

First example: Elastic scattering $^{12}C+p$

Data from H.O. Meyer et al., Z. Phys. A279 (1976) 41





R matrix fits for different channel radii

a	E_R	Γ	E_0	$\gamma_0 2$	χ^2
4.5	0.4273	0.0341	-1.108	1.334	2.338
5	0.4272	0.0340	-0.586	1.068	2.325
5.5	0.4272	0.0338	-0.279	0.882	2.321
6	0.4271	0.0336	-0.085	0.745	2.346

→ E_R, Γ very stable with a

→ global fit independent of a

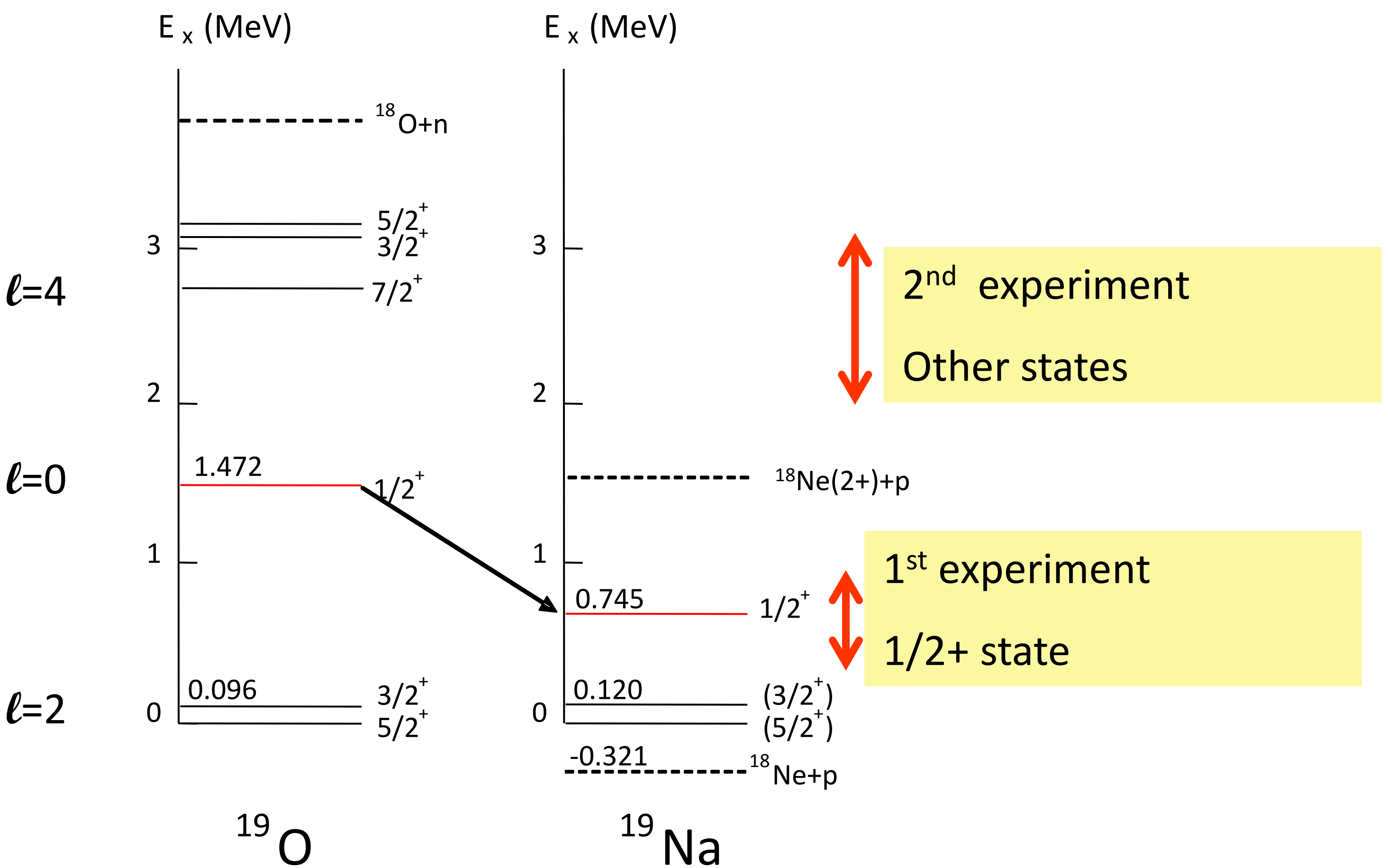
4. Applications to elastic scattering

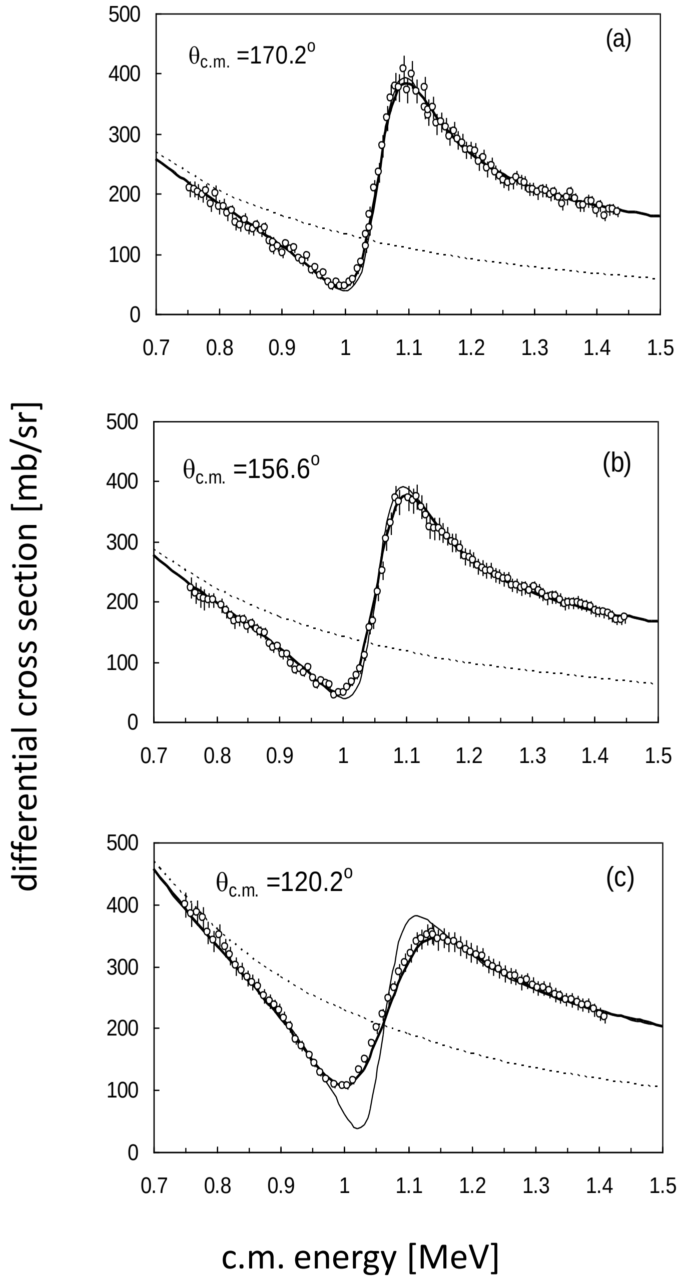
Second example: ^{18}Ne+p scattering at Louvain-la-Neuve

First Experiment : ^{18}Ne+p elastic: *C. Angulo et al, Phys. Rev. C67 (2003) 014308*

→ search for the mirror state of ^{19}O(1/2$^+$)

Second experiment: ^{18}Ne(p,p')^{18}Ne(2$^+$): *M.G. Pellegriti et al, PLB 659 (2008) 864*



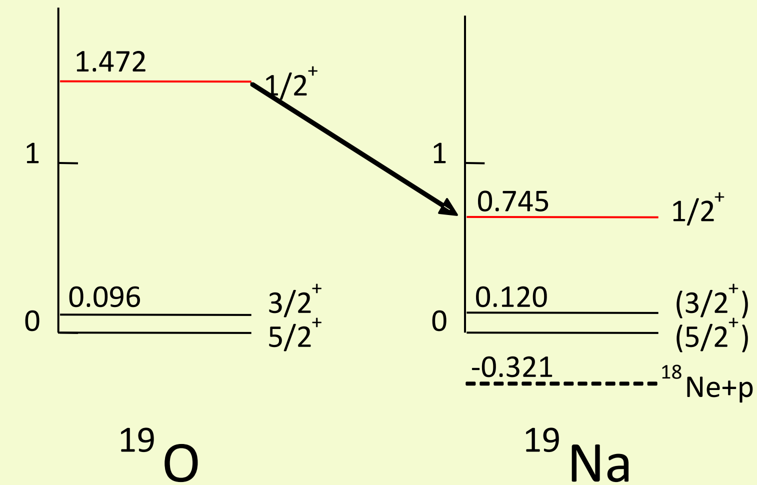


^{18}Ne+p elastic scattering

Final result

E_R = 1.066 ± 0.003 MeV

Γ_p = 101 ± 3 keV

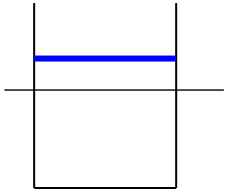


→ Very large Coulomb shift

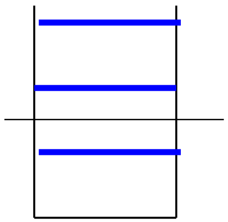
From Γ=101 keV, γ^2=605 keV, θ^2=23%

Very large reduced width

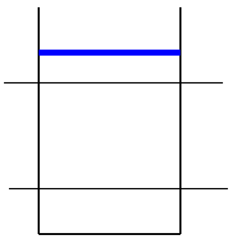
="single-particle state"



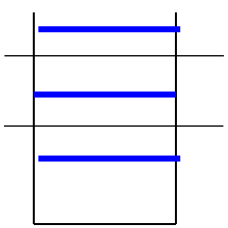
- **one channel, one pole** :elastric scattering only (isolated resonance)
- $R(E) = \frac{\gamma_0^2}{E_0 - E}$



- **One channel, several poles**: elastic scattering (more than 1 resonance)
- $R(E) = \sum_{\lambda=1}^{N} \frac{\gamma_\lambda^2}{E_\lambda - E}$



- **Several channels, one pole** : transfer or inelastic cross sections (isolated resonance)
- In practice: 2 channels (but could be larger)
- $R_{ij}(E) = \frac{\gamma_i \gamma_j}{E_0 - E}$



- **Several channels, several poles**: most complicated
- $R_{ij}(E) = \sum_\lambda \frac{\gamma_i^\lambda \gamma_j^\lambda}{E_\lambda - E}$

Extension to one-pole, two channels (transfer reactions)

Properties of the pole: energy E_0

reduced width in the entrance channel γ_i

reduced width in the exit channel γ_i

R-matrix: 2x2 matrix

$R_{ii}(E) = \frac{\gamma_i^2}{E_0 - E}$ associated with the entrance channel

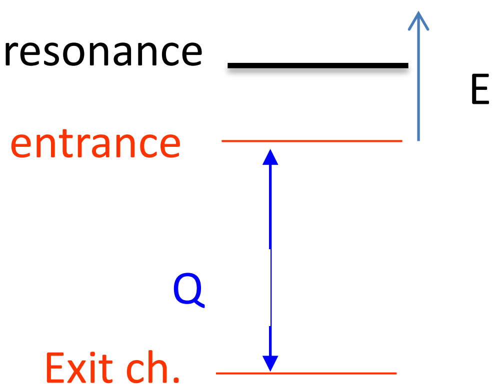
$R_{ff}(E) = \frac{\gamma_f^2}{E_0 - E}$ associated with the exit channel

$R_{if}(E) = \frac{\gamma_i \gamma_f}{E_0 - E}$ associated with the transfer

Total width $\Gamma(E) = \Gamma_i(E) + \Gamma_f(E)$ with

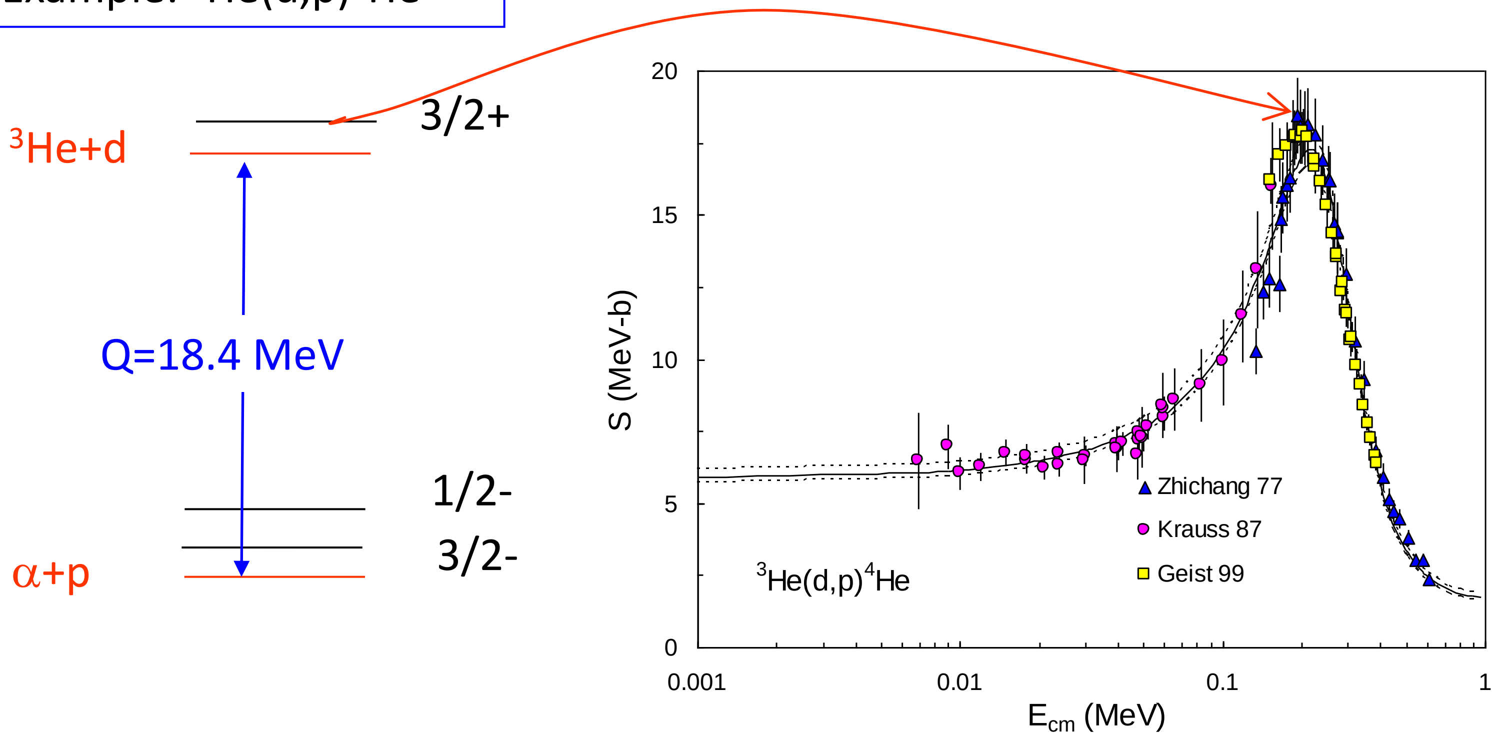
$$\Gamma_i(E) = 2\gamma_i^2 P_i(E)$$

$$\Gamma_f(E) = 2\gamma_f^2 P_f(E + Q)$$



- Previous formula can be easily extended
- ℓ values can be different in both channels
- From 2x2 R-matrix → 2x2 scattering matrix U
- U_{11}, U_{22} → elastic cross sections
- U_{12},→transfer cross section

Example: $^{3}He(d,p)^{4}He$



3/2+ resonance:

- Entrance channel: spin S=1/2,3/2, parity + → ℓ=0,2
- Exit channel: spin S=1/2, parity + → ℓ=1

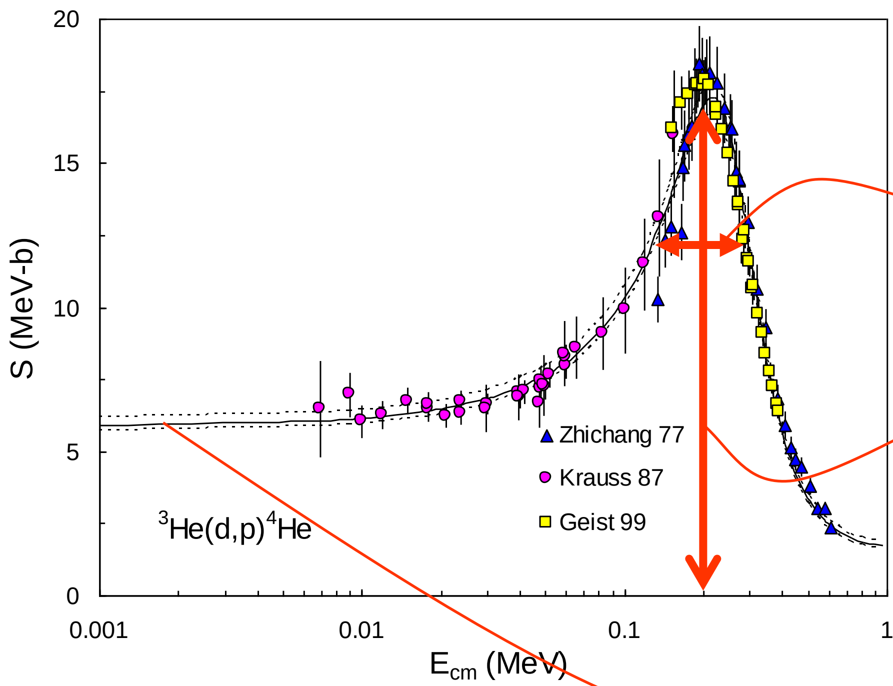
Breit Wigner approximation

$$\sigma_{ij}(E) \approx \frac{\pi}{k^2} \frac{2J+1}{(2I_1+1)(2I_2+1)} \frac{\Gamma_i(E)\Gamma_f(E)}{(E-E_R)^2+\Gamma^2/4}$$

with

$$\Gamma_i(E) = 2\gamma_i^2 P_{\ell_i}(E)$$

$$\Gamma_f(E) = 2\gamma_f^2 P_{\ell_f}(E+Q)$$ Nearly constant if Q is large



Width at half maximum=total width Γ

Amplitude: ~$\Gamma_i\Gamma_f/\Gamma^2$

$$\sigma_{ij}(E) \sim \frac{1}{E}\Gamma_i(E) \sim \frac{1}{E} P_{\ell_i}(E)$$

$$\rightarrow S(E) \sim S_0 \text{ if } \ell_i = 0$$

5. Extension to inelastic scattering and transfer

^{18}Ne(p,p')^{18}Ne(2$^+$) inelastic scattering

Combination of ^{18}Ne(p,p)^{18}Ne elastic and ^{18}Ne(p,p')^{18}Ne(2$^+$) inelastic

➔ constraints on the R-matrix parameters

Generalization to 2 channels: $R_{ij}(E) = \frac{\gamma_i \gamma_j}{E_0 - E}$

i=1: ^{18}Ne(0$^+$)+p channel

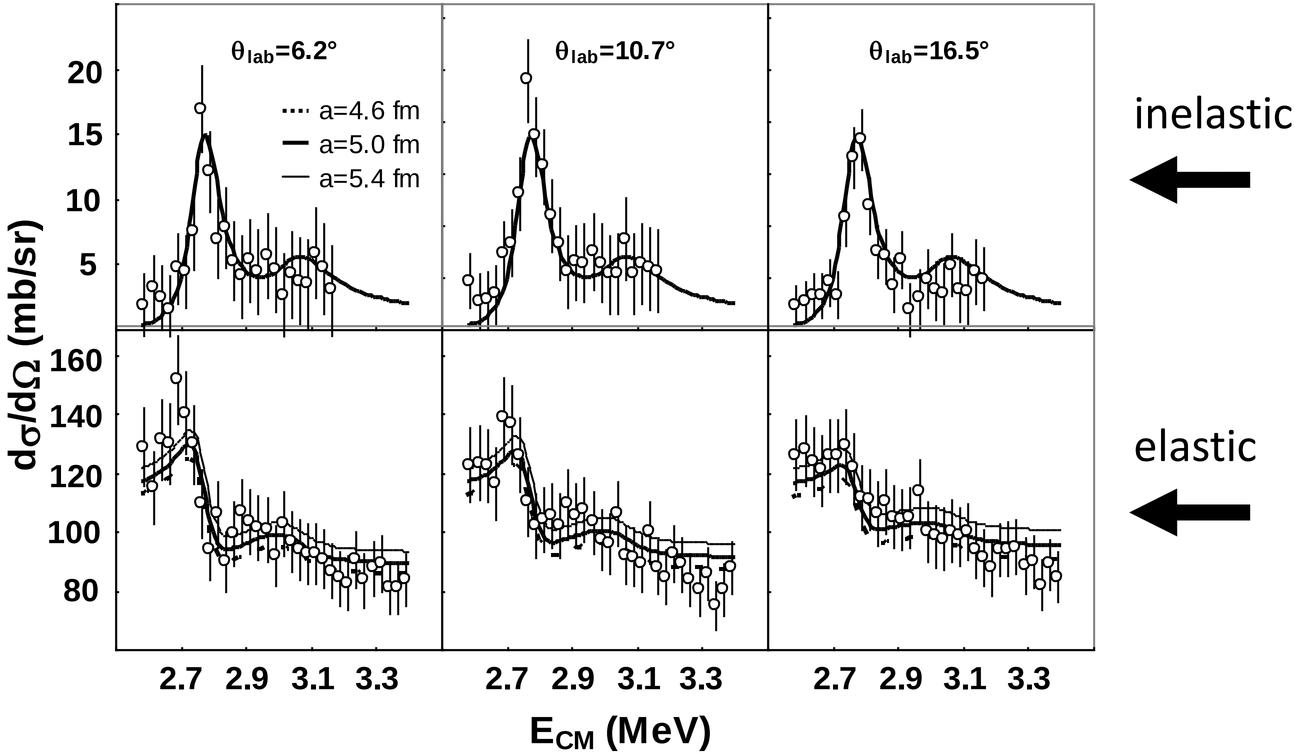
i=2: ^{18}Ne(2$^+$)+p channel

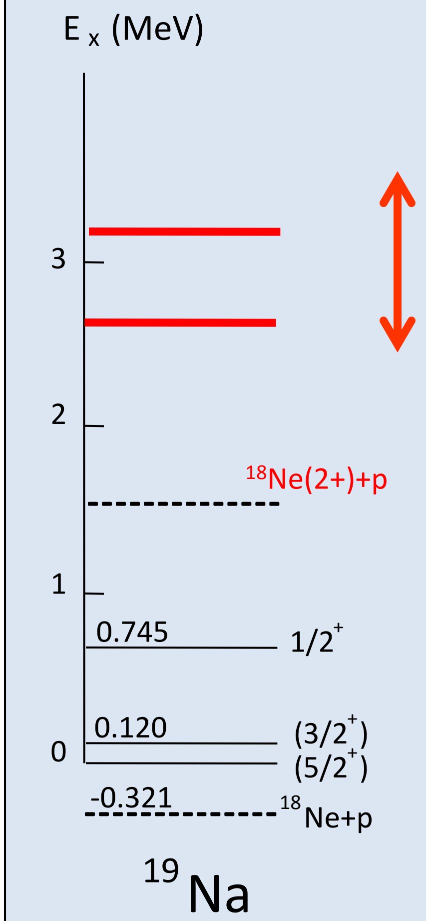
➔ each state has 3 parameters: E_0, γ_1, γ_2

From R-matrix → collision matrix U_{ij}
Elastic cross section obtained from U_{11}
Inelastic cross section obtained from U_{12}

6. Phenomenological R matrix Method

^{18}Ne+p inelastic scattering: *M.G. Pellegriti et al, PLB 659 (2008) 864*





Several angles fitted simultaneously

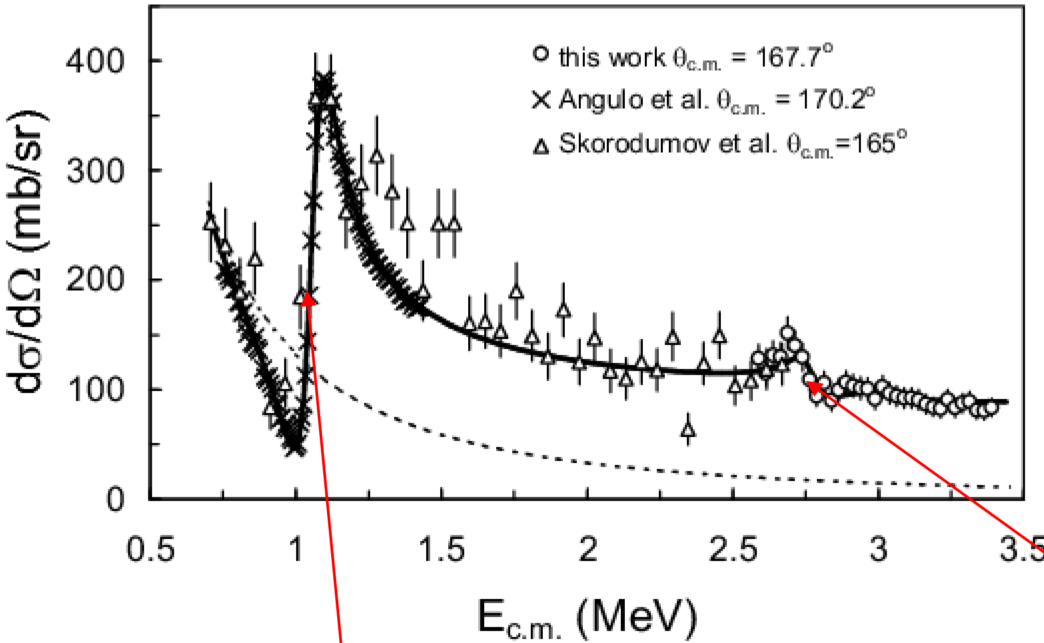
Presence of 2 states → 6 parameters

$E_{\rm c.m.}$ (MeV)	$2J^\pi$	$\Gamma_{\rm tot}$ (keV)	$(2J+1)\frac{\Gamma_0}{\Gamma_{\rm tot}}$	θ_0^2 (%)	θ_2^2 (%)
2.78 ± 0.03	$(5,3)^+$	105 ± 10	0.43 ± 0.05	1.1 ± 0.3	44 ± 4
3.09 ± 0.06	$(3,5)^+$	250 ± 50	0.12 ± 0.04	0.6 ± 0.2	36 ± 7

→dominant p+^{18}Ne(2$^+$) structure

6. Phenomenological R matrix Method

^{18}Ne+p elastic scattering: comparison with other experiments



Complete set of data up to 3.5 MeV

θ_0 large ~23%

→ observable in elastic scattering

→ dominant p+^{18}Ne(0$^+$) structure

→ single-particle state

θ_2 large (2 states) ~40%, θ_0 small

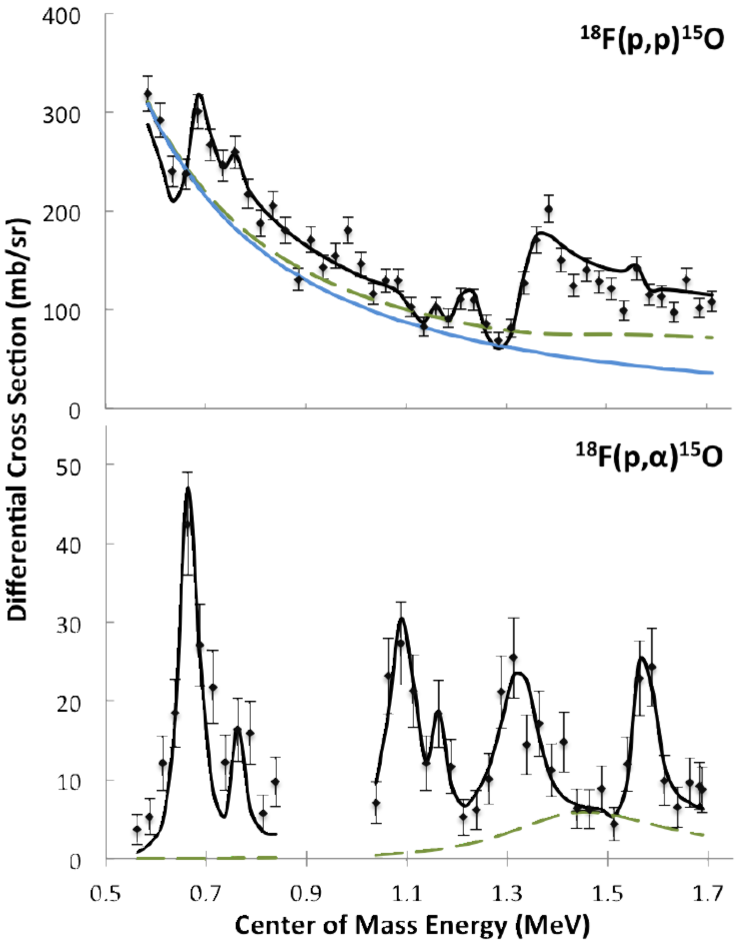
→ difficult to observe in elastic scattering

→ dominant p+^{18}Ne(2$^+$) structure

→ single-particle states with excited core

6. Phenomenological R matrix Method

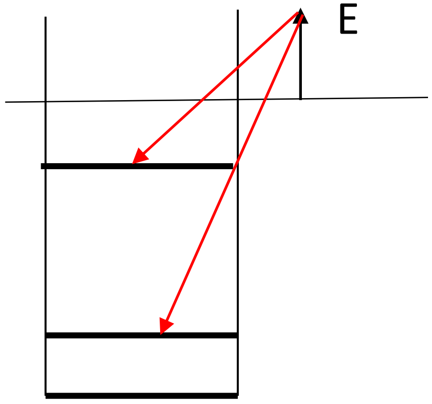
Recent application to $^{18}F(p,p)^{18}F$ and $^{18}F(p,\alpha)^{15}O$
D. Mountford et al, to be published



simultaneous fit of both cross sections
angle: 176°
for each resonance: $J\pi, E_R, \Gamma_p, \Gamma_\alpha$
8 resonances → 24 parameters

6. Phenomenological R matrix Method

Radiative capture



Capture reaction=transition between an initial state at energy E to bound states

Cross section $\sigma_C(E) \sim \left|< \Psi_f \middle| H_\gamma \middle| \Psi_i(E) >\right|^2$

Additional pole parameter: gamma width $\Gamma_{\gamma i}$

$$< \Psi_f \middle| H_\gamma \middle| \Psi_i(E) > = < \Psi_f \middle| H_\gamma \middle| \Psi_i(E) >_{int} + < \Psi_f \middle| H_\gamma \middle| \Psi_i(E) >_{ext}$$

internal part: $< \Psi_f | H_\gamma | \Psi_i(E) >_{int} \sim \sum_{i=1}^{N} \frac{\gamma_i \sqrt{\Gamma_{\gamma i}}}{E_i - E}$

external part:

$$< \Psi_f | H_\gamma | \Psi_i(E) >_{ext} \sim C_f \int_a^\infty W(2k_f r) r^\lambda (I_i(kr) - U O_i(kr)) dr$$

More complicated than elastic scattering!

But: many applications in nuclear astrophysics

7. Conclusions

1. One R-matrix for each partial wave (limited to low energies)
2. Consistent description of resonant and non-resonant contributions (not limited to resonances!)
3. The R-matrix method can be applied in two ways
   a) Calculable R-matrix: to solve the Schrödinger equation
   b) Phenomenological R-matrix: to fit experimental data (low energies, low level densities)

4. Calculable R-matrix
   - Application in many fields of nuclear and atomic physics
   - Efficient to get phase shifts and wave functions of scattering states
   - 3-body systems
   - Stability with respect to the radius is an important test

5. Phenomenological R-matrix
   - Same idea, but the pole properties are used as free parameters
   - Many applications: elastic scattering, transfer, capture, beta decay, etc.